Worked solutions to Unit 7 (Differential Equations) problems from notes

1. C. Translating the description of the relationship, "the rate of change of the quantity of with respect to the quantity of present" is $\frac{dS}{dT}$. "Inversely proportional to" means one quantity is proportional to the *reciprocal* of the other, like time and speed in traveling a given distance: speed $= \frac{\text{distance}}{\text{time}}$. That's where all the fractions come from in the choices. The constant, like distance, is in the numerator. And then "the natural logarithm of the quantity of *T*" gives the denominator; that's $\frac{dS}{dT} = \frac{k}{\ln T}$.

2. C. "The population increases with respect to time at a rate that is directly proportional to the population" means we want something like $\frac{dP}{dt} = kP$, which narrows the choices to C and D. Since the population's increase of 2 million is one-fifth of the 10 million current population of insects, this is C. Notice that A isn't a differential equation at all, but it *is* a tangent line to the function.

3. C. The fact that the volume decreases as pressure increases means the rate must be negative, so it's not A or B. "The rate of change of the volume of gas with respect to the pressure" is $\frac{dV}{dP}$.

4. A. If the given function is a solution to the differential equation, then we can substitute it in to find an equation to solve. That will require finding y''. Differentiating, $y' = \frac{d}{dx}\left(e^{-2x} + ke^{4x}\right) = -2e^{-2x} + 4ke^{4x}$. So $y'' = \frac{d}{dx}\left(-2e^{-2x} + 4ke^{4x}\right) = 4e^{-2x} + 16ke^{4x}$. Substituting, $y - \frac{y''}{4} = 5e^{4x}$ becomes $e^{-2x} + ke^{4x} - \frac{4e^{-2x} + 16ke^{4x}}{4} = 5e^{4x}$. The left side simplifies nicely: $e^{-2x} + ke^{4x} - \frac{4e^{-2x} + 16ke^{4x}}{4} = e^{-2x} + ke^{4x} - e^{-2x} - 4ke^{4x} = -3ke^{4x}$. Then $-3ke^{4x} = 5e^{4x}$, and $k = \frac{5}{-3}$.

5. C. Like the previous problem, the idea is to work out y'' and substitute into the differential equation. $y' = 5k\cos(5x) - 8\sin(4x)$, and then $y'' = -25k\sin(5x) - 32\cos(4x)$. Making the substitution into the differential equation, $y'' + 16y = -27\sin(5x)$ becomes $-25k\sin(5x) - 32\cos(4x) + 16\left(k\sin(5x) + 2\cos(4x)\right) = -27\sin(5x)$. Then this:
$-25k\sin(5x) - 32\cos(4x) + 16k\sin(5x) + 32\cos(4x) = -27\sin(5x)$
$-9k\sin(5x) = -27\sin(5x)$
$k = \frac{-27\sin(5x)}{-9\sin(5x)} = 3$

6. C. Yes, this is a lot of C answers. The questions came from many different years, so I wouldn't read too much into that. The differential equation $\frac{dy}{dx} = e^{2x} + 2y$ isn't separable; it can be solved

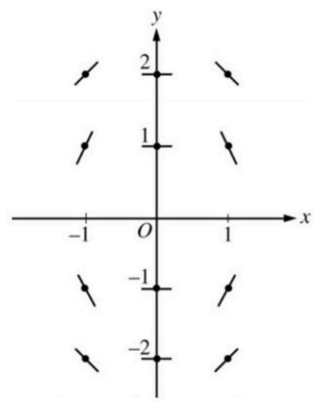
analytically, but not with a technique that you've learned in this course. That's okay — one of the choices is right, so we can just check them. The first two choices are easier to differentiate (and you can eliminate them easily if you know that $\frac{dy}{dx} = ky$ is the differential equation that leads to exponential functions). Checking each of the choices:

A: If $y = e^{2x}$, then $\frac{dy}{dx} = 2e^{2x}$, and substituting into the differential equation looks like $2e^{2x} = e^{2x} + 2 \cdot e^{2x} = 3e^{2x}$. That's not true, so it's not A.

B: Same deal, with a negative. If $y = -e^{2x}$, then $\frac{dy}{dx} = -2e^{2x}$, and substituting into the differential equation looks like $-2e^{2x} = e^{2x} + 2 \cdot -e^{2x} = -e^{2x}$. That's false, so it's not B.

C: This time the derivative needs the product rule. If $y = xe^{2x}$, then $\frac{dy}{dx} = x \cdot 2e^{2x} + e^{2x} \cdot 1$, and substituting into the differential equation gives $2xe^{2x} + e^{2x} = e^{2x} + 2 \cdot xe^{2x}$. That one is true. No need to check D.

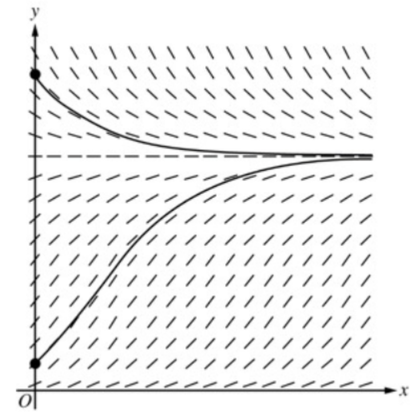
7. (a) At the locations where $x = 0$, on the y-axis, we have $\frac{dy}{dx} = -\frac{2 \cdot 0}{y} = 0$ and horizontal tangent segments. When $y = 1$, the derivative is just $-2x$, so we get slopes of 2 and –2 at the points (–1, 1) and (1, 1), respectively. When $y = -1$, the slopes work out to $2x$, so that gives slopes of –2 and 2 at the points (–1, –1) and (1, –1). Then for $y = 2$, the slopes are $-\frac{2x}{2} = -x$, so at (–1, 2) and (1, 2), the slopes are 1 and –1, and the opposites of those where $y = -2$. You can see all of this in the graph on the right.



(b) A tangent line requires a point and a slope. The point is given as (1, –1), and the slope there is 2, as calculated in part (a). That makes the required tangent line $y - (-1) = 2(x - 1)$. To approximate $f(1.1)$, substitute that value of x and solve the line for y: $f(1.1) \approx 2(1.1 - 1) + (-1)$. No need to simplify, but if you do, that's –0.8.

(c) Now we solve the differential equation. First separate the variables, then integrate: $\frac{dy}{dx} = -\frac{2x}{y}$ gives $y\,dy = -2x\,dx$. Then $\int y\,dy = \int -2x\,dx$, and $\frac{1}{2}y^2 = -x^2 + C$. At this point we'll both need to solve for y to get the $y = f(x)$ form required and find the value of C. I've come to believe that it's usually wise to find C first, based on how the markschemes are written. Substituting the point from the initial condition gives $\frac{1}{2}(-1)^2 = -(1)^2 + C$, and then $C = \frac{3}{2}$. So now we have $\frac{1}{2}y^2 = -x^2 + \frac{3}{2}$. Multiplying by 2, we get $y^2 = -2x^2 + 3$. The quadratic has two solutions, positive and negative, but we're looking for a function, which means we have to decide whether + or – is the right sign to use. The initial condition, (1, –1), answers that question. The only y-coordinate we have is negative, so we must use the negative result: $y = -\sqrt{-2x^2 + 3}$.

8. (a) Obviously, your graphs have to go through the indicated points. They need to follow the slopes, and there is a horizontal asymptote. The scoring guide for the problem says this: "Each solution curve must pass through the appropriate point, extend reasonably close to the right edge of the square, and have no obvious conflicts with the given slope lines." That's what they're looking for. When "reasonably close" has been explained by the graders, it generally means that you didn't stop before the last column of slopes.



(b) The differential equation given is continuous for $y > 0$, so this limit can be found by direct substitution:

$\lim_{y \to 50} \left(-\frac{y}{4} \ln\left(\frac{y}{50}\right)\right) = -\frac{50}{4} \ln\left(\frac{50}{50}\right)$. You can leave it like that, but since $\ln 1 = 0$ the limit evaluates to 0.The markscheme points out lots of incorrect notation involving linkage errors that gets penalized. Be very sure to write what you mean.

(c) Since $f(0) = 20, f'(0) = \left.\frac{dy}{dx}\right|_{(0,\,20)} = -\frac{20}{4} \ln\left(\frac{20}{50}\right)$. Since $\frac{20}{50} < 1$,

$f'(0) = -\frac{20}{4} \ln\left(\frac{20}{50}\right) = \text{negative} \cdot \text{negative} > 0$. The value of $f'(x) > 0$ for all $0 < y < 50$, so f is increasing, and since we already know that the slope is 0 when $y = 50$, the function won't cross the line $y = 50$ as $x \to \infty$. That means f must be increasing for all x. (Happily, this matches the shape of the slope field and the lower of the two solution curves.)

(d) To find the maximum of $\frac{dy}{dx}$, we check its derivative, which is $\frac{d^2y}{dx^2}$, to look for critical points. The open interval given means that there are no endpoints.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{y}{4} \ln\left(\frac{y}{50}\right)\right) = -\frac{y}{4} \cdot \left(\frac{1}{\frac{y}{50}} \cdot \frac{1}{50} \cdot \frac{dy}{dx}\right) + \ln\left(\frac{y}{50}\right) \cdot -\frac{1}{4}\frac{dy}{dx}$$

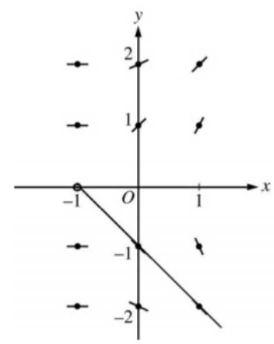
$= -\frac{1}{4} \cdot \frac{dy}{dx}\left(y \cdot \frac{1}{y} + \ln\left(\frac{y}{50}\right)\right) = -\frac{1}{4} \cdot \frac{dy}{dx}\left(1 + \ln\left(\frac{y}{50}\right)\right)$. We already know that $\frac{dy}{dx} = 0$ for $y = 50$, which isn't on the specified interval, so $\frac{d^2y}{dx^2} = 0$ when $1 + \ln\left(\frac{y}{50}\right) = 0$, so $\ln\left(\frac{y}{50}\right) = -1$, and changing to exponential form, $e^{-1} = \frac{y}{50}$. That makes the critical value $y = 50e^{-1}$. Checking the signs of $\frac{d^2y}{dx^2}$, when $0 < y < 50e^{-1}$, $\frac{d^2y}{dx^2} < 0$, and for $y > 50e^{-1}$, $\frac{d^2y}{dx^2} > 0$. That means $\frac{dy}{dx}$ has a maximum when $y = 50e^{-1}$.

9. E. First the product rule gives $h'(x) = 3x \cdot f'(x) + f(x) \cdot 3$. Then $h'(-1) = 3(-1) \cdot f'(-1) + f(-1) \cdot 3$. The value of $f'(-1)$ can be read off the diagram by counting the slope there as –2. That gives $h'(-1) = 3(-1) \cdot -2 + 2 \cdot 3 = 6 + 6 = 12$.

10. D. The slope field shows that $\frac{dy}{dx} = 0$ for $y = 4$. Substituting those values into the differential equation, we have $0 = k \cdot 4 - 2 \cdot 4^2 = 4k - 32$, so $k = 8$.

11. C. The slope field shows that y' is undefined when $x = -1$. That eliminates choice A, which is defined everywhere. Choice B is not equal to 0 when $x = 0$, so that's out, too. Choice D, based on tangent, would give undefined values of y when $x = \frac{\pi}{2}$ and everywhere else that tan x has a vertical asymptote. The slope field does not show that, so it can't be right. That leaves C.

12. (a) The slopes will be zero when the numerator of $\frac{dy}{dx}$ is 0, so when $x = -1$. That gives the column of horizontal slopes on the left. Here are the rest.



$$\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{1}{2} \qquad \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1}{1} = 1 \qquad \left.\frac{dy}{dx}\right|_{(0,-1)} = \frac{1}{-1} = -1$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2}{2} = 1 \qquad \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2}{1} = 2 \qquad \left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{2}{-1} = -2$$

$$\left.\frac{dy}{dx}\right|_{(0,-2)} = \frac{1}{-2} = -\frac{1}{2} \qquad \left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{2}{-2} = -1$$

The solution curve starts at the point (0, –1) and the slope there of –1 leads to the graph you see. The derivative is undefined at $y = 0$, so there's an open circle at that point, and the solution cannot continue above the x-axis.

(b) When $\frac{dy}{dx} = \frac{x+1}{y} = -1$, we have $-y = x + 1$ and therefore $y = -x - 1$. Therefore $\frac{dy}{dx} = -1$ for all points on the line $y = -x - 1$, except where $y = 0$.

(c) First separate the variables, then integrate: $\int y\,dy = \int (x+1)dx$ gives $\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C$. Using the initial condition, $\frac{1}{2}(-2)^2 = \frac{1}{2}(0)^2 + 0 + C$, and $C = 2$. Then $\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + 2$ is multiplied by 2 to give the quadratic relationship $y^2 = x^2 + 2x + 4$. Because the initial condition has a negative y-coordinate, $y = -\sqrt{x^2 + 2x + 4}$.

13. This appears in more than one of these files (and it's not the only problem that does), because it includes topics from more than one unit. It is one of the "everything" problems from the May 2020 pandemic online exam, hence the seven parts. Barring another emergency shutdown, I doubt you'll ever see more than four parts in a real free response question.

(a) Average rate of change needs a difference quotient, and by now you probably recall that they're looking to see both the difference and the quotient, but that no simplification is needed. $M'(6) = \frac{M(18) - M(0)}{18 - 0} = \frac{46 - 40}{18 - 0}$. The units are $\frac{\text{units of } M}{\text{units of } t} = \frac{\text{coins per day}}{\text{day}}$.

(b) This is an instantaneous rate of change, and that has to be clear in the explanation: $M'(6)$ is represents the rate at which the rate of change of the number of coins is increasing on day 6.

The markscheme specifically says they're looking to see the word "rate" twice here. The day can be referred to with "at" or "on" "day 6" or "the sixth day," and there must be a reference to the number of coins. I recognize that the wording is extremely awkward. They listed several different ways it might be worded, and *all* of them seem awkward to me.

(c) For a tangent line, we need a point and a slope. For the tangent to $y = C(t)$ at $t = 6$, the point will be $\left(6, C(6)\right)$, or (6, 300). The slope is $C'(6) = M(6) = 54$, from the table. The line is therefore $y - 300 = 54(t - 6)$. Before you ask, they did accept x in place of t there and $C(t)$ in place of y, but the way I typed it uses the preferred variables.

(d) To find $\int_0^6 M'(3t)dt$, let $u = 3t$, so that $du = 3dt$. When $t = 0$, $u = 0$ for the lower limit; when $t = 6$, $u = 18$ for the upper limit. Multiplying and dividing by 3 to get the expression for du into the integral and substituting gives us $\frac{1}{3}\int_0^6 M'(3t)\cdot 3dt = \frac{1}{3}\int_0^{18} M'(u)du$. Then the antiderivative is easy. $\frac{1}{3}\left[M(u)\right]_0^{18} = \frac{1}{3}\left(M(18) - M(0)\right) = \frac{1}{3}(46 - 40) = 2$.

(e) The area of a trapezoid can be calculated with the formula $A = \frac{1}{2}h(b_1 + b_2)$, where the height is the horizontal distance between the two vertical bases. You can draw a graph if you like, but I'm not going to. $\int_0^{18} M(t)dt \approx \frac{1}{2}\cdot(6 - 0)\cdot(40 + 54) + \frac{1}{2}\cdot(18 - 6)\cdot(54 + 46)$.

(f) For each additional coin collected, the total weight increases by 3 grams. Therefore the rate at which the weight changes, in total, is three times the rate of change of total number of coins for both classes. That's $3\left(M(7) + B(7)\right)$.

(g) How does this part connect to what came before? I have no idea. I think they were just trying to get a slope field into one of the two long free-response questions that constituted 100% of the online exam that year. The differential equation $\frac{dy}{dt} = 0.3y$ gives slopes that only depend on the value of y; we should see the same slope all the way across each different horizontal set. But we don't; at $y = 1$, for instance, the slopes start at 0 on the y-axis and increase as we go to the right. This can't be the right slope field.
It's also possible to recognize that $\frac{dy}{dt} = 0.3y$ is a special case of $\frac{dy}{dt} = ky$, and the solution of that is one you've seen many times before: $y = Ce^{kt}$. That resulting exponential function must have a horizontal asymptote on the x-axis, but we can see that this slope field has non-zero slopes along the horizontal axis; it cannot be the slope field for our differential equation. That was my first thought, but I decided that noticing slopes only depend on y is both easier and more generally applicable to this sort of question.

14. A. The ln 2 is just a constant here, so $\int \frac{1}{x\ln 2}dx = \frac{1}{\ln 2}\int \frac{1}{x}dx = \frac{1}{\ln 2}\ln|x| + C = \frac{\ln|x|}{\ln 2} + C$. The solution of a differential equation can't cross the place where $\frac{dy}{dx}$ is undefined, so either $x > 0$ or $x < 0$. If $x > 0$, then $\frac{\ln|x|}{\ln 2} + C = \frac{\ln x}{\ln 2} + C$.

15. B. First, separate the variables and integrate to get $\int \cos y dy = \int \cos x e^{\sin x} dx$. The left side evaluates to $\sin y$, but the right side would benefit from u-substitution. Let $u = \sin x$, so that $du = \cos x dx$. That gives $\sin y = \int e^u du = e^u + C$. If we go back to the original variable, we have $\sin y = e^{\sin x} + C$. We generally isolate y in these things, and all of the choices have been solved for y. That takes an inverse sine, so $y = \arcsin\left(e^{\sin x} + C\right)$.

16. B. Separating the variables and setting up the integrals gives $\int 4y dy = \int x \cos(x^2) dx$. The left side is a straightforward power rule, and the right side can be integrated by using u-substitution, with $u = x^2$ and $du = 2x dx$. That gives $2y^2 = \frac{1}{2}\int 2x \cos(x^2) dx = \frac{1}{2}\int \cos(u) du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C$. At this point, you ought to be able to see why B is the correct choice, but the remaining steps are to divide by 2, giving $y^2 = \frac{1}{4}\sin(x^2) + C_2$, and then to take the square root. Since we know $y > 0$, the positive version is our choice: $y = \sqrt{\frac{1}{4}\sin(x^2) + C_2}$.

17. (a) First we separate the variables and integrate: $\int e^{2y} dy = \int 3x^2 dx$. This time it's the left side that would benefit from u-substitution, but you might do without it. I'll let $u = 2y$ so that $du = 2dy$. Then $\frac{1}{2}\int e^{2y} \cdot 2dy = \int 3x^2 dx$, so $\frac{1}{2}\int e^u du = x^3 + C$, and that left side comes out to $\frac{1}{2}e^u = \frac{1}{2}e^{2y}$, so we have $\frac{1}{2}e^{2y} = x^3 + C$. The initial condition of $f(0) = \frac{1}{2}$ can be substituted to get $\frac{1}{2}e^{2\cdot\frac{1}{2}} = 0^3 + C$, and $C = \frac{1}{2}e$ means that $\frac{1}{2}e^{2y} = x^3 + \frac{1}{2}e$. The question asks for "a solution $y = f(x)$," so we still have to solve for y. Multiplying by 2 gives $e^{2y} = 2x^3 + e$. Taking logarithms on both sides, we have $2y = \ln(2x^3 + e)$, and then $y = \frac{1}{2}\ln(2x^3 + e)$. This part of the question was worth six marks of the nine in the problem.

(b) For the domain, we need the argument of the logarithm to be positive: $2x^3 + e > 0$, so $x^3 > -\frac{1}{2}e$, and $x > \sqrt[3]{-\frac{1}{2}e}$. Ugly, but the cube root of a negative is fine. Since the argument of the logarithm can be any real number, and the range of the natural logarithm is all reals, $y \in (-\infty, \infty)$.

18. (a) If the horizontal line $y = -2$ is tangent to the graph, then we're looking for a place where $\frac{dy}{dx} = 0$. If $\frac{3 - x}{y} = 0$, then $x = 3$. To determine if this point (3, –2) is a maximum, minimum, or neither is going to require the second derivative test.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3-x}{y}\right) = \frac{y(-1)-(3-x)\cdot\frac{dy}{dx}}{y^2}$. Evaluating that at (3, –2), we have $\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \frac{-2(-1)-(3-3)\cdot 0}{(-2)^2} = \frac{2}{4} = \frac{1}{2}$. Since the curve is concave up at this location of a horizontal tangent, the graph has a minimum there.

(b) Once again, it is necessary to separate the variables and integrate: $\int y\,dy = \int (3-x)dx$, and this time we only need the power rule to find that $\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$. Since $g(6) = -4$, this gives $\frac{1}{2}(-4)^2 = 3(6) - \frac{1}{2}\cdot 6^2 + C$, and then $C = 8$, so $\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + 8$. Finally, solving for y requires multiplication by 2 and then taking the square root: $y = \pm\sqrt{6x - x^2 + 16}$. This time the initial condition has a negative y-coordinate, so we need the negative version, and $y = -\sqrt{6x - x^2 + 16}$.

19. A. Integrating to find y, we have $y = \int 16\sin^3 x\cos x\,dx$. That can be evaluated with u-substitution, with $u = \sin x$ and $du = \cos x\,dx$. Then $y = \int 16u^3du = 16\cdot\frac{1}{4}u^4 + C$ $= 4\sin^4 x + C$. Using the information that $y = 16$ when $x = \frac{\pi}{2}$ gives $16 = 4\left(\sin\frac{\pi}{2}\right)^4 + C = 4 + C$, and thus $C = 12$. Now we have $y = 4\sin^4 x + 12$, so if $x = \frac{\pi}{6}$, $y = 4\left(\sin\frac{\pi}{6}\right)^4 + 12 = 4\cdot\left(\frac{1}{2}\right)^4 + 12 = \frac{1}{4} + \frac{48}{4} = \frac{49}{4}$.

20. D. Separating the variables gives $\int y^{-2}dy = \int dx$, so $-y^{-1} = x + C$. It might be easiest to see this as a fraction, $-\frac{1}{y} = x + C$. Substituting the initial condition gives $-1 = 1 + C$, so $C = -2$. That means $-\frac{1}{y} = x - 2$. Multiplying by –1, we get $\frac{1}{y} = -x + 2$, and then $y = \frac{1}{-x+2}$. This narrows the choices down to C and D. The domain is determined by the fact that the solution of a differential equation cannot cross a place where the derivative or the function itself is undefined. Ours is undefined at $x = 2$, and the initial condition uses $x = 1$, to the left of that, so we pick the one with $x < 2$.

21. C. The key to separating the variables here is rewriting e^{y+x} as e^ye^x, which is the $a^ma^n = a^{m+n}$ law of exponents in reverse. Then $\int \frac{1}{e^y}dy = \int e^xdx$. The right side is simple, and the left side can be written as a negative exponent: $\int e^{-y}dy = \int e^xdx$, so $-e^{-y} = e^x + C$. As we know $y(0) = -\ln 4$, that gives $-e^{-(-\ln 4)} = e^0 + C$. The left side is a bit messy, but $-e^{-(-\ln 4)} = -e^{\ln 4} = -4$, and then

$-4 = 1 + C$, so $C = -5$. Putting that back in the general solution, $-e^{-y} = e^x - 5$ gives $e^{-y} = -e^x + 5$, then $-y = \ln\left(-e^x + 5\right)$. Finally, we have $y = -\ln\left(-e^x + 5\right)$. In fact, once I saw that C was 5, I might have skipped to that choice.

22. A. The basic form of the differential equation, $y' = ky$, means we're looking at an exponential function, and only A and B are candidates; the initial condition only satisfies A. However, I know there are some people who would not recognize that basic form (sigh), so I will work it out. Separate the variables and integrate: $\int \frac{1}{y} dy = \int 4dx$. Then the left side is a logarithm and the right side is a power rule: $\ln\left|y\right| = 4x + C$. The positive value of y in the initial condition means the absolute value signs are unnecessary: $\ln y = 4x + C$. Exponentiating, $y = e^{4x+C}$. And finally using laws of exponents there gives $y = e^{4x+C} = e^{4x}e^{C} = C_2 e^{4x}$. Like I said, that's exponential. The initial condition substitutes in to give $4 = C_2 e^{4 \cdot 0} = C_2 \cdot 1 = C_2$, so we have $y = 4e^{4x}$.

23. D. The first sentence describes the differential equation $\frac{dA}{dt} = kA$, where I'm using A to represent the *amount* of chemical remaining. That has the general solution $A = Ce^{kt}$. (Since I just solved a very similar one in #22, I won't do that here.) The initial amount of 12 moles at $t = 0$ means $C = 12$. The second fact, that there are 4 moles left at time $t = 4$, will allow us to determine the value of k: $4 = 12e^{k \cdot 4}$, so $e^{4k} = \frac{1}{3}$, and $k = \frac{1}{4}\ln\left(\frac{1}{3}\right)$. That makes the model $A = 12e^{\frac{1}{4}\ln\left(\frac{1}{3}\right)t}$. When there are 3 moles of chemical remaining, that equation becomes $3 = 12e^{\frac{1}{4}\ln\left(\frac{1}{3}\right)t}$. First we divide by 12, then take logarithms to get $\frac{1}{4} = e^{\frac{1}{4}\ln\left(\frac{1}{3}\right)t}$ and then $\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln\left(\frac{1}{3}\right)t$. Multiplying by 4 and dividing by the $\ln\left(\frac{1}{3}\right)$ leaves us with $t = \frac{4\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{1}{3}\right)}$. Unfortunately, that doesn't look exactly like any of the choices. (This seems really algebra-heavy to me, but it is a question from AP Classroom.) The key here is to use exponents and laws of logs. $\frac{4\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{1}{3}\right)} = \frac{4\ln(4^{-1})}{\ln(3^{-1})} = \frac{-4\ln 4}{-\ln 3} = \frac{4\ln 4}{\ln 3}$. I think it's pretty unlikely that you'd need this much algebra on a multiple-choice question on the exam. It's more likely you'd have a calculator to get a decimal both for k and for the time at the end.

24. C. The first differential equation, $\frac{dP}{dt} = -0.4P$, leads to the solution $P = Ce^{-0.4t}$, and its initial condition of $P(0) = 2$ means $P = 2e^{-0.4t}$. The second differential equation is, as the problem says, linear; if $\frac{dQ}{dt} = -1$, then $Q = -t + C_2$, and its initial condition of $Q(0) = 3$ means $C_2 = 3$, so $Q = -t + 3$. Those are the same size when $2e^{-0.4t} = -t + 3$, which requires a calculator to solve. My calculator gets $t = 2.156$.

25. This is another repeat from an earlier set.

(a) The point we need is given as (12, 36). The slope will be $\left.\dfrac{dB}{dt}\right|_{t=12} = \dfrac{5}{\sqrt{12+4}} + \dfrac{1}{9}(12-6)^2 = \dfrac{5}{4} + \dfrac{1}{9} \cdot 36 = 1.25 + 4 = 5.25$. It's not required to work that out, but I think it's a bit easier to deal with. That makes the required tangent line $y - 36 = 5.25(t - 12)$. We approximate the value as $B(10) \approx 5.25(10 - 12) + 36$ hundred books. Note that the "hundred" there is important. If you work out the arithmetic, you get $-10.5 + 36 = 25.5$ hundred, or 2550 books. The markscheme actually gives the answer as 2550.

(b) $B''(t) = \dfrac{d}{dt}\left(\dfrac{dB}{dt}\right) = \dfrac{d}{dt}\left(\dfrac{5}{\sqrt{t+4}} + \dfrac{1}{9}(t-6)^2\right) = \dfrac{d}{dt}\left(5(t+4)^{-1/2} + +\dfrac{1}{9}(t-6)^2\right)$ $= -\dfrac{5}{2}(t+4)^{-3/2} \cdot 1 + \dfrac{2}{9}(t-6) \cdot 1$, so $B''(5) = -\dfrac{5}{2} \cdot 9^{-3/2} + \dfrac{2}{9} \cdot -1$. That value is clearly negative, which is important to the interpretation. Five hours after the books go on sale, the rate at which books are sold is decreasing at this many hundred books per hour per hour. (Yes, it's lazy to say "this many," but it's truly not necessary to write that expression again, or to work it out. Also, notice the second "per hour" in the units.)

(c) For $B(t)$, we integrate. $B(t) = \displaystyle\int \left(5(t+4)^{-1/2} + \frac{1}{9}(t-6)^2\right) dt$ $= 5 \cdot \dfrac{(t+4)^{1/2}}{1/2} + \dfrac{1}{9} \cdot \dfrac{(t-6)^3}{3} + C$. The initial condition of $B(12) = 36$ allows us to find that constant: $36 = 5 \cdot \dfrac{(12+4)^{1/2}}{1/2} + \dfrac{1}{9} \cdot \dfrac{(12-6)^3}{3} + C = 5 \cdot 2 \cdot \sqrt{16} + \dfrac{6^3}{9 \cdot 3} + C$ $= 40 + \dfrac{6 \cdot 6 \cdot 6}{3 \cdot 3 \cdot 3} + C = 40 + 2 \cdot 2 \cdot 2 + C = 48 + C$, so $C = -12$. That makes the model $B(t) = 5 \cdot \dfrac{(t+4)^{1/2}}{1/2} + \dfrac{1}{9} \cdot \dfrac{(t-6)^3}{3} - 12$, and the predicted number of books sold by $t = 21$ equal to $B(21) = 5 \cdot \dfrac{25^{1/2}}{1/2} + \dfrac{1}{27} \cdot 15^3 - 12$ hundred books. Again, no need to work that out. The markscheme tells me it's 16,300 books.