

Worked solutions to Unit 6 (Integration and Accumulation of Change) problems from notes

1. B. Checking them in turn: Since $f'(x) > 0$ for $0 < x < 1$, $f(1) > f(0)$, so I is false. Since $f'(x) > 0$ for $1 < x < 2$, $f(2) > f(1)$, so II is true. Since $\int_1^3 f'(x)dx > 0$, $f(3) > f(1)$, and III is false.
2. A. We have the rate of change and the initial temperature, so the final temperature is the initial temperature plus the integral of the rate of change: $200 + \int_0^4 R'(t)dt \approx 175.165$.
3. The squirrel changes direction when $v(t)$ changes signs. We can just calculate the position every time that happens. The position at $t = 0$ is at building A, so the distance from A is 0. At time $t = 9$, the position is $0 + \int_0^9 v(t)dt = \frac{1}{2} \cdot 20(9 + 5) = 140$. At time $t = 15$, the position is $0 + \int_0^{15} v(t)dt = 140 + -\frac{1}{2} \cdot 10(6 + 4) = 90$. Finally, at $t = 18$, the position is $0 + \int_0^{18} v(t)dt = 90 + \frac{1}{2} \cdot 10(3 + 2) = 115$. So the squirrel is farthest from A at $t = 9$, and the distance is 140.
4. D. Since there can be any number of subintervals, the easiest number to use is just one. The left Riemann sum will have a height of 0, and that's obviously an underestimate. The right sum will more than include the entire region, and it is an overestimate. Because the graph is concave down, a trapezoid will be under the curve, and the second underestimate.
5. C. The interval is 2 units wide, and with four subintervals, each of those is 0.5 units wide. The right Riemann sum will use x -coordinates of 0.5, 1, 1.5, and 2. The approximation is $\int_0^2 9^x dx \approx 0.5 \cdot 9^{0.5} + 0.5 \cdot 9^1 + 0.5 \cdot 9^{1.5} + 0.5 \cdot 9^2 = 0.5(3 + 9 + 27 + 81) = 0.5 \cdot 120 = 60$.
6. B. Option I is a left Riemann sum. Since $f(x)$ is increasing, that will be an underestimate, so I is true. Option II is a midpoint Riemann sum. Without knowing more about the shape of the curve, it's not possible to know whether this is an under- or overestimate; concavity matters. Option III is a right Riemann sum. Since $f(x)$ is increasing, this one is an overestimate, and III is true.
7. C. Since the graph of R is concave down, trapezoids will give an underestimate. Draw a sketch to see it.
8. A. The trapezoidal sum is an overestimate, so the curve must be concave up. That eliminates B, C, and E. The right Riemann sum is an underestimate, so the curve must be decreasing. That eliminates D.

9. D. The formula for the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$.

$$\int_0^6 f(x)dx \approx \frac{1}{2} \cdot 2 \cdot (4+k) + \frac{1}{2} \cdot 2 \cdot (k+8) + \frac{1}{2} \cdot 2 \cdot (8+12) = 32 + 2k$$

Since $32 + 2k = 52$, $k = 10$.

10. (a) Average acceleration is $\frac{\Delta v}{\Delta t} = \frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80 - 0}$ feet per second per second.
 (b) The integral of velocity is change in position, so the integral represents the change in the rocket's position, in feet, from $t = 10$ to $t = 80$ seconds. (Change in position is synonymous with displacement. Since velocity is known to be positive, this can also be interpreted as distance traveled.) The midpoint Riemann sum will be

$$\int_{10}^{80} v(t)dt \approx 20 \cdot v(20) + 20 \cdot v(40) + 20 \cdot v(60) = 20(22 + 35 + 44).$$

- (c) We know that the velocity of Rocket A is 49 ft/s at $t = 80$. To calculate the velocity of Rocket B, add the initial velocity to the integral of the acceleration from $t = 0$ to $t = 80$:

$$\begin{aligned} 2 + \int_0^{80} 3(t+1)^{-1/2} dt &= 2 + [2 \cdot 3(t+1)^{1/2}]_0^{80} \\ &= 2 + (6\sqrt{81} - 6\sqrt{1}) = 2 + 54 - 6 = 50. \end{aligned}$$

That means Rocket B is traveling one foot per second faster at $t = 80$.

11. (a) The tangent line has slope $r'(5) = 2.0$ from the table and $r(5) = 30$ from the given radius. The tangent line is $y - 30 = 2.0(t - 5)$, so $r(5.4) \approx 2.0(5.4 - 5) + 30$. Since we know r is concave down on the relevant interval, the tangent line will give an overestimate, so this is greater than the true value.

- (b) Related rates! Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$. Then $\frac{dV}{dt} \Big|_{t=5} = 4\pi \cdot 30^2 \cdot 2.0$ cubic feet per minute.

- (c) The table has the y -values we need for the Riemann sum.

$$\int_0^{12} r'(t)dt \approx (2-0)(4.0) + (5-2)(2.0) + (7-5)(1.2) + (11-7)(0.6) + (12-11)(0.5).$$

While you can do the subtractions in your head if you want, they're going to need to see the sum of five products to get you the points here. During the interval from $t = 0$ to $t = 12$ minutes, the radius of the balloon has increased by approximately this many feet. (Again, notice my "this many" reference to avoid either calculating that or writing it again.)

- (d) Since r is concave down, $r'(t)$ is decreasing on the interval, and a right Riemann sum will be an underestimate; the approximation is less than the true value of the integral.

12. A. The $\frac{5}{n}$ term tells that the width of the interval is 5 units, but all of the choices are five units wide, so that's no help. Since a definite integral is the limit of a Riemann sum as the number of subintervals goes to infinity, consider what the first and last terms of the sum look like as $n \rightarrow \infty$.

$\lim_{n \rightarrow \infty} \ln \left(2 + \frac{5}{n} \cdot 1 \right) = \ln 2$ and $\lim_{n \rightarrow \infty} \ln \left(2 + \frac{5}{n} \cdot n \right) = \ln 7$. That suggests that the first rectangle has a height — a y -value — of $\ln 2$ and the last a height of $\ln 7$. The integrand in choice A has those values at the limits of integration. (By comparison, choice B would give $\ln 4$ and $\ln 9$, choice C would give $\ln 2$ and $\ln 27$, and choice D would give $\ln 12$ and $\ln 37$.)

13. (a) We're given that $h(2) = 7$, and the table indicates that $h'(2) = 0.3$, so the tangent line is $y - 7 = 0.3(t - 2)$. Then $h(2.8) \approx 0.3(2.8 - 2) + 7$. Since we know $h(t)$ is concave up, the tangent line lies below the curve, so this will be an underestimate.
- (b) The right Riemann sum uses the heights starting at $t = 4$:
- $$\int_2^{14} h'(t) dt \approx (4 - 2) \cdot 0.6 + (5 - 4) \cdot 0.7 + (8 - 5) \cdot 1.4 + (10 - 8) \cdot 2.3 + (14 - 10) \cdot 7.7.$$
- The units are $\frac{\text{feet}}{\text{hour}} \cdot \text{hours} = \text{feet}$.
- (c) Because $h(t)$ is concave up, $h''(t) > 0$, and therefore $h'(t)$ must be increasing. With an increasing function, a right Riemann sum will be an overestimate.
- (d) The $\frac{15}{n}$ indicates that the interval is 15 units wide, divided into n pieces. We know that $n \rightarrow \infty$. If $k = 1$, for the first rectangle, the height of that rectangle will be $h' \left(\frac{15}{n}(1 - 1) \right) = h'(0)$, for a lower limit of integration of 0. As we know the width of the interval to be 15, that should make the upper limit 15, of course, but checking with the right-most rectangle where $k = n$, that height looks like $h' \left(\frac{15}{n}(n - 1) \right) = h' \left(15 - \frac{15}{n} \right)$, and as $n \rightarrow \infty$, that does go to $h'(15)$. Therefore the integral is $\int_0^{15} h'(t) dt$.
14. D. The $\frac{5}{n}$ means that the interval is 5 units wide, but all of the choices share that. The left-most rectangle has $k = 1$, which means the first height is $\frac{10}{n} \sqrt{1 + \frac{5}{n}}$, and as $n \rightarrow \infty$, that expression goes to 0, which gives a lower limit of integration of 0. The choices are now between integrands of $10\sqrt{1 + x}$ and $2x\sqrt{1 + x}$. The coefficient of that radical depends on k , which changes, so the choice is $2x\sqrt{1 + x}$. Notice that $\frac{10k}{n} \left(\sqrt{1 + \frac{5k}{n}} \right) = 2 \cdot \frac{5}{n} \cdot k \left(\sqrt{1 + \frac{5}{n} \cdot k} \right)$. Each of the $\frac{5}{n}$ terms represents x , which further backs up the $2x$ coefficient and the $1 + x$ radicand.
15. E. It is easiest to see that II is a correct integral; it has a width of 5 from the $\frac{5}{n}$ on the end, and an argument of $-1 + x$ accompanies that. Option I is a translation of that one unit to the left, and therefore it has the same value. If I and II are both true, the choices require that III works, too. It's harder to see why that ought to be right. The interval for III is only one unit wide — but the vertical dilation caused by the coefficient of 5 outside the integral counteracts the horizontal

dilation by a factor of $\frac{1}{5}$ that the coefficient of x in the argument of the sine. Therefore this integral must have the same value as the others. These questions are difficult, and I wouldn't expect more than one appearance on the AP calculus exam.

16. E. This is an application of the Fundamental Theorem of Calculus, the part that says $\frac{d}{dx} \int_a^x f(t) dt = f(x)$. Because the upper limit of integration is not just x , we need the chain rule. That x^3 in the upper limit of integration is the “inside” part. So the x^3 takes the place of t in the integrand, and we multiply by its derivative: $\frac{d}{dx} \left(\int_0^{x^3} \ln(t^2 + 1) dt \right) = \ln((x^3)^2 + 1) \cdot 3x^2$.

17. C. By the FTC, $f'(x) = e^{(x^2 - 3x)^2} \cdot (2x - 3)$, using the chain rule — the same process as in the previous problem. Since the $e^{(x^2 - 3x)^2}$ is always positive, the value of $f'(x) = 0$ only for $x = \frac{3}{2}$. There are no endpoints, as $f(x)$ is defined for all real numbers. Checking the signs, $f'(x) < 0$ for $x < \frac{3}{2}$, and $f'(x) > 0$ for $x > \frac{3}{2}$, so $x = \frac{3}{2}$ must produce a minimum.

18. C. Once again using the Fundamental Theorem to differentiate the integral function, $g'(x) = f(2x) \cdot 2$. Then $g'(3) = f(6) \cdot 2 = -1 \cdot 2 = -2$. That value of $f(6)$ is read directly from the given graph.

19. (a) We know that $f(t)$ has a domain of $[0, 5]$. The upper limit of integration of $h(x)$ must be an input of f . If $t = 0$, then $\frac{x}{2} + 3 = 0$, and $x = -6$. On the other end of the domain of f , if $t = 5$, then $\frac{x}{2} + 3 = 5$, and $x = 4$. That makes the domain of $h(x)$ come out to $[-6, 4]$.

- (b) Once again we differentiate the integral function using the FTC and the chain rule: $h'(x) = f\left(\frac{x}{2} + 3\right) \cdot \frac{1}{2}$. Then $h'(2) = f\left(\frac{2}{2} + 3\right) \cdot \frac{1}{2} = f(4) \cdot \frac{1}{2} = -\frac{3}{2}$.

- (c) For a minimum, we consider critical points and endpoints. $h'(x) = 0$ when $f\left(\frac{x}{2} + 3\right) = 0$. Since the graph shows $f(2) = 0$, $\frac{x}{2} + 3 = 2$, and $x = -2$ is the only critical point of h .

Because the sign of $h'(x)$ is the same as the sign of f , the change of sign of $f(x)$ at its zero from positive to negative means that $x = -2$ is a local maximum of $h(x)$, not a local minimum. The maximum must either be at $x = -6$ or at $x = 4$, as those are the two endpoints for $h(x)$. It's possible (and easy) to calculate the value at the left endpoint:

$$h(-6) = \int_0^{-\frac{6}{2}+3} f(t) dt = \int_0^0 f(t) dt = 0. \text{ Without a formula for the function, though, we}$$

can't get an exact value at the right endpoint. However, $h(4) = \int_0^{\frac{4}{2}+3} f(t) dt = \int_0^5 f(t) dt$.

Based on the diagram — with its very helpful grid — we can see that this integral is negative. Therefore $h(4) < h(-6)$, and the minimum occurs at $x = 4$. This is a fairly old question. These days, I suspect the function would be piecewise with shapes whose areas you could actually calculate to get numbers here.

20. D. $g(5) = \int_0^5 f(t)dt = \frac{1}{2} \cdot 1 \cdot 3 + \frac{1}{2}\pi \cdot 2^2 = \frac{3}{2} + 2\pi$. Geometry.

21. (a) First, $g(-6) = \int_{-2}^{-6} f(t)dt$. Since those limits of integration run right to left, it might be easier to deal with this if we use the property of definite integrals that says reversing the limits of integration changes the sign of the result: $g(-6) = - \int_{-6}^{-2} f(t)dt = -\frac{1}{2} \cdot 4 \cdot 5$.

Then $g(3) = \int_{-2}^3 f(t)dt = \frac{1}{2}\pi \cdot 2^2 + \frac{1}{2} \cdot 1 \cdot -2$, where that negative “height” is because the triangle on the right end is below the x -axis.

(b) By the FTC, $g'(x) = f(x)$, so $g'(0) = f(0) = 2$.

(c) The graph of $g(x)$ will have a horizontal tangent where $g'(x) = f(x) = 0$, at $x = -2$ and $x = 2$. (The question refers to the *open* interval, which is why -6 isn't included in this list.) At $x = -2$, $g' = f$ does not change signs, so g has neither a maximum nor a minimum. At $x = 2$, $g' = f$ changes signs from positive to negative, and g has a local maximum.

(d) Most of the time we think of points of inflection as where the second derivative, g'' , changes signs, but that's equivalent to where g' has a local maximum or minimum. Since we have the graph of $g' = f$, we can see those local extrema of g' are at $x = -4$, $x = -2$, and $x = 0$.

22. A. We will find a relative maximum of g where $g' = f$ changes signs from positive to negative. That's only at $x = 1$. At $x = 3$, the negative-to-positive sign change gives a local minimum for g . At $x = 4$ and $x = 5$, g' has a local max, but that's not the same as g having one. Finally, at $x = 6$, $g'(x) = 0$, but its sign does not change, and g has neither a maximum nor minimum there.

23. B. Starting at the left side of the interval, $g(x)$ will be zero when the upper limit is -5 , since $\int_{-5}^{-5} f(t)dt = 0$. As we move to the right, the positive and negative regions between $x = -5$ and $x = -2$ will exactly cancel each other out, so we get a second zero of $x = -2$. And then the symmetric positive and negative regions between $x = -2$ and $x = 6$ give a third zero at $x = 6$.

24. D. Both I and II can be determined by looking at $g'(x) = f(x)$. Since $g' = f > 0$ on the given interval, g must be increasing there. And if g is increasing on the whole interval, then it cannot have a relative maximum on the open interval — the maximum would have to be at the right endpoint. So I is true and II is false. For III, $g(x) = 0$ when $x = 1$, since that gives

$$g(1) = \int_1^1 f(t)dt = 0, \text{ and III is true.}$$

25. C. When $f'(x) > 0$, $f(x)$ is increasing, and when $f'(x) < 0$, $f(x)$ is decreasing. I'll call the x -intercepts of $f'(x)$ between 0 and 10 by the names a and b , from left to right. Because $f(0) = 2$, $f(a) = 2 + 20 = 22$, $f(b) = 22 - 6 = 16$, and $f(10) = 16 + 4 = 20$. The greatest of these is 22.
26. D. By the Fundamental Theorem of Calculus, $g'(x) = \sin(x^2)$. Based on a graph, $g'(x) = \sin(x^2) \leq 0$ for $1.77 \leq x \leq 2.51$.
27. (a) Relative extrema only happen where at critical points, and those are at $x = 1$ and $x = 2$. At $x = 1$, f' does not change signs, so that is neither. At $x = 2$, f' changes signs from positive to negative, producing a local maximum.
- (b) The graph of f will start with the four given points: $(0, -1)$, $(1, 0)$, $(2, 2)$, and $(3, 0)$. Since $f'(1) = 0$, f will have a horizontal tangent at $(1, 0)$. We already know that $f(2)$ is a maximum, and the nonexistent derivative there means a corner. It's not a cusp because the second derivative changes signs at $x = 2$, and it's not a vertical tangent line because the location is a maximum. Then getting the concavity to be down on $0 < x < 1$, up on $1 < x < 2$, down on $2 < x < 3$, and up on $3 < x < 4$ gives the graph you see here.
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- (c) Since $g(x) = \int_1^x f(t)dt$, $g'(x) = f(x)$. The graph of $g(x)$ will have a relative maximum where $g'(x) = f(x)$ changes signs from positive to negative, at $x = 3$, and g will have a relative minimum where $g' = f$ changes signs from negative to positive, at $x = 1$.
- (d) The graph of g will have a point of inflection when $g' = f$ has a local maximum or minimum, at $x = 2$.
28. E. Fundamental Theorem of Calculus again, but this time with the chain rule.
 $f'(x) = \sqrt{(2x)^2 - (2x)} \cdot 2$, so $f'(2) = \sqrt{4^2 - 4} \cdot 2 = 2\sqrt{12}$.

29. D. FTC with the chain rule. $\frac{dy}{dx} = (3(x^2)^2 - 5(x^2)^2) \cdot 2x = (3x^4 - 5x^2) \cdot 2x = 6x^5 - 10x^3$
30. (a) Since $g(x) = \int_0^x f(t)dt$, we know that $g'(x) = f(x)$ and $g''(x) = f'(x)$. Counting squares and making the matching triangles above and below the x -axis cancel out, $g(4) = \int_0^4 f(t)dt = 3$. Then $g'(4) = f(4) = 0$, and counting the slope as rise over run gives $g''(4) = f'(4) = -2$.
- (b) Relative extrema can be determined from the sign changes of the derivative. Since $g' = f$ changes signs from negative to positive at $x = 1$, this is the location of a relative minimum.
- (c) Since the period is 5, every 5 units will add the same amount to the area. With the symmetric pieces above and below the horizontal axis, we can use geometry (counting squares, really)

to see that $\int_0^5 f(t)dt = 2$. If $g(5) = 2$, then $g(10) = g(5) + \int_5^{10} f(t)dt = 2 + 2 = 4$. The closest we get to $x = 108$ counting by fives is 105, and that's 21 sets of 5. The last bit, $\int_{105}^{108} f(t)dt = \int_0^3 f(t)dt = 2$. That means $g(108) = \int_0^{108} f(t)dt = 21 \cdot 2 + 2 = 44$. The slope there is $g'(108) = g'(3) = f(3) = 2$. And finally the tangent line is $y - 44 = 2(x - 108)$.

31. A. We have $\int_3^{10} (f(x) - g(x)) dx$, and we want $\int_0^3 (f(x) - g(x)) dx$. If we knew $\int_0^{10} (f(x) - g(x)) dx$, we'd be able to subtract $\int_0^{10} (f(x) - g(x)) dx - \int_3^{10} (f(x) - g(x)) dx$ to get the desired value. The two other integrals will let us figure that out. $\int_0^{10} (f(x) - g(x)) dx = \int_0^{10} f(x)dx - \int_0^{10} g(x)dx$. We don't quite have the integral of g that needs, but almost; if $\int_0^{10} \frac{1}{2}g(x)dx = 8$, then $\int_0^{10} g(x)dx = 16$. Then $\int_0^{10} (f(x) - g(x)) dx = 21 - 16 = 5$, and $\int_0^{10} (f(x) - g(x)) dx - \int_3^{10} (f(x) - g(x)) dx = 5 - 2 = 3$.

32. B. Since by additivity of integrals $\int_{-1}^{10} g(x)dx = \int_{-1}^3 g(x)dx + \int_3^{10} g(x)dx$, and $\int_{-1}^{10} g(x)dx = -\int_{10}^{-1} g(x)dx = -12$, we have $-12 = \int_{-1}^3 g(x)dx + \int_3^{10} g(x)dx$. To get that integral on the interval $[-1, 3]$ takes the use of some other properties: $\int_{-1}^3 (2g(x) + 4) dx = 2 \int_{-1}^3 g(x)dx + \int_{-1}^3 4dx = 2 \int_{-1}^3 g(x)dx + 16 = 22$. Subtracting the 16 to move it to the other side and then dividing by 2 gives $\int_{-1}^3 g(x)dx = 3$, and finally $-12 = 3 + \int_3^{10} g(x)dx$, so $\int_3^{10} g(x)dx = -15$. That took a lot more work than I anticipated.

33. B. This one is pretty straightforward, but there are a lot of little pieces to combine. Here are the separate integrals, one at a time.

$$\int_{-4}^{-2} f(x)dx = 2 \cdot 1 = 2$$

$$\int_{-2}^2 f(x)dx = 4 \cdot 1 + \frac{1}{2}\pi \cdot 2^2 = 4 + 2\pi$$

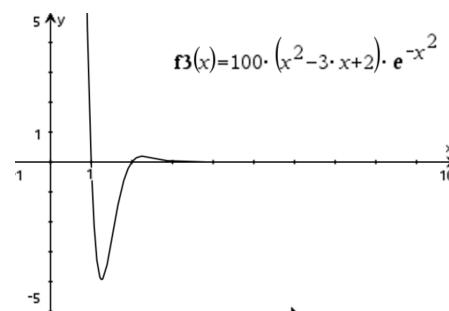
$$\int_2^3 f(x)dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

$$\int_3^5 f(x)dx = -\frac{1}{2}\pi \cdot 1^2 = -\frac{1}{2}\pi$$

$$\int_5^7 f(x)dx = \frac{1}{2} \cdot 2 \cdot -1 = -1$$

Adding all of those gives

$$2 + 4 + 2\pi + \frac{1}{2} - \frac{1}{2}\pi - 1 = \frac{11}{2} + \frac{3\pi}{2}.$$



34. C. $\int_{-3}^3 (f(x) + 1)dx = \int_{-3}^3 f(x)dx + \int_{-3}^3 1 dx$. The first of those is $-2 + 2 + (-2) = -2$. The second is 6, because it's a rectangle one unit tall and six units wide. The total is 4.

35. C. First substituting for f , then antidifferentiating, $\int_1^3 f(x)dx = \int_1^3 g'(x)dx = [g(x)]_1^3 = g(3) - g(1) = (3^2 - 3 \cdot 3 + 4) - (1^2 - 3 \cdot 1 + 4) = 2$.

36. C. Straightforward use of the FTC to evaluate the definite integral:

$$\int_{-2}^2 (4x^3 + 6x) dx = [x^4 + 3x^2]_{-2}^2 = (2^4 + 3 \cdot 2^2) - ((-2)^4 + 3(-2)^2) = 0.$$

37. D. There are a couple of ways to do this. One of them is to write an equation for that line — after all, you can see the y -intercept and count the slope — and then find the function f . I choose to use the relationship $\text{final} = \text{initial} + \int_a^b \text{rate of change } dx$ instead. The values of a and b are 0 and 1, respectively.

$$\text{So } f(1) = f(0) + \int_0^1 f(x)dx = 5 + \frac{1}{2} \cdot 1 \cdot 6 = 8.$$

38. A. I'll just integrate: $\int_{-3}^k x^2 dx = \left[\frac{1}{3}x^3\right]_{-3}^k = \frac{1}{3}k^3 - \frac{1}{3}(-3)^3 = \frac{1}{3}k^3 + 9$. If that's equal to zero, then $k^3 = -27$, and $k = -3$.

39. A. No tricks: $\int_2^x (3t^2 - 1) dt = [t^3 - t]_2^x = (x^3 - x) - (8 - 2) = x^3 - x - 6$.

40. B. To check whether $g(x)$ is increasing, we need $g'(x)$. The FTC gives $g'(x) = 100(x^2 - 3x + 2)e^{-x^2}$. A graph of that shows that $g'(x) < 0$ on $(1, 2)$, so option I is false, but $g'(x) > 0$ on $(2, 3)$, so II is true. To check option III, the calculator can evaluate that integral, $g(3) = \int_1^3 100(t^2 - 3t + 2)e^{-t^2} dt \approx -1.942$, which is not positive. Only II is true.

41. A. This is another final = initial + \int_a^b rate of change dx question: $f(5) = f(2) + \int_2^5 f'(x)dx$. Using the calculator, that's $f(5) \approx 0.4 + 28.605 = 29.005$.
42. E. Using the integral evaluation part of the FTC gives that $\int_0^4 f'(x)dx = [f(x)]_0^4 = f(4) - f(0)$.
Substituting the values we know, then, $8 = [f(x)]_0^4 = f(4) - 5$, and $f(4) = 13$.
43. Yes, this appears in more than one of these files (and it's not the only problem that does), because it includes topics from more than one unit. It is one of the "everything" problems from the May 2020 pandemic online exam, hence the seven parts. Barring another emergency shutdown, I doubt you'll ever see more than four parts in a real free response question.
- (a) Average rate of change needs a difference quotient, and by now you probably recall that they're looking to see both the difference and the quotient, but that no simplification is needed. $M'(6) = \frac{M(18) - M(0)}{18 - 0} = \frac{46 - 40}{18 - 0}$. The units are $\frac{\text{units of } M}{\text{units of } t} = \frac{\text{coins per day}}{\text{day}}$.
- (b) This is an instantaneous rate of change, and that has to be clear in the explanation: $M'(6)$ is represents the rate at which the rate of change of the number of coins is increasing on day 6. The markscheme specifically says they're looking to see the word "rate" twice here. The day can be referred to with "at" or "on" "day 6" or "the sixth day," and there must be a reference to the number of coins. I recognize that the wording is extremely awkward. They listed several different ways it might be worded, and *all* of them seem awkward to me.
- (c) For a tangent line, we need a point and a slope. For the tangent to $y = C(t)$ at $t = 6$, the point will be $(6, C(6))$, or $(6, 300)$. The slope is $C'(6) = M(6) = 54$, from the table. The line is therefore $y - 300 = 54(t - 6)$. Before you ask, they did accept x in place of t there and $C(t)$ in place of y , but the way I typed it uses the preferred variables.
- (d) To find $\int_0^6 M'(3t)dt$, let $u = 3t$, so that $du = 3dt$. When $t = 0$, $u = 0$ for the lower limit; when $t = 6$, $u = 18$ for the upper limit. Multiplying and dividing by 3 to get the expression for du into the integral and substituting gives us $\frac{1}{3} \int_0^6 M'(3t) \cdot 3dt = \frac{1}{3} \int_0^{18} M'(u)du$. Then the antiderivative is easy. $\frac{1}{3} [M(u)]_0^{18} = \frac{1}{3} (M(18) - M(0)) = \frac{1}{3}(46 - 40) = 2$.
- (e) The area of a trapezoid can be calculated with the formula $A = \frac{1}{2}h(b_1 + b_2)$, where the height is the horizontal distance between the two vertical bases. You can draw a graph if you like, but I'm not going to. $\int_0^{18} M(t)dt \approx \frac{1}{2} \cdot (6 - 0) \cdot (40 + 54) + \frac{1}{2} \cdot (18 - 6) \cdot (54 + 46)$.
- (f) For each additional coin collected, the total weight increases by 3 grams. Therefore the rate at which the weight changes, in total, is three times the rate of change of total number of coins for both classes. That's $3(M(7) + B(7))$.
- (g) How does this part connect to what came before? I have no idea. I think they were just trying to get a slope field into one of the two long free-response questions that constituted 100% of

the online exam that year. The differential equation $\frac{dy}{dt} = 0.3y$ gives slopes that only depend on the value of y ; we should see the same slope all the way across each different horizontal set. But we don't; at $y = 1$, for instance, the slopes start at 0 on the y -axis and increase as we go to the right. This can't be the right slope field.

It's also possible to recognize that $\frac{dy}{dt} = 0.3y$ is a special case of $\frac{dy}{dt} = ky$, and the solution of that is one you've seen many times before: $y = Ce^{kt}$. That resulting exponential function must have a horizontal asymptote on the x -axis, but we can see that this slope field has non-zero slopes along the horizontal axis; it cannot be the slope field for our differential equation. That was my first thought, but I decided that noticing slopes only depend on y is both easier and more generally applicable to this sort of question.

44. B. Just the power rule for integration with the FTC:

$$\int_{-1}^3 (x^2 - 2x) dx = \left[\frac{1}{3}x^3 - x^2 \right]_{-1}^3 = \left(\frac{1}{3} \cdot 27 - 9 \right) - \left(\frac{1}{3} \cdot -1 - 1 \right) = 0 + \frac{4}{3}.$$

45. E. This might look a bit like u -substitution, but it's not, because we never want du in the denominator. Instead, separate the integrand into two fractions and deal with them separately.

$$\int \frac{e^x + 4}{e^x} dx = \int \left(\frac{e^x}{e^x} + \frac{4}{e^x} \right) dx = \int (1 + 4e^{-x}) dx = x - 4e^{-x} + C.$$

46. A. Integrate, one step at a time. First, $f'(x) = \int (2x - \cos x) dx = x^2 - \sin x + C_1$. Then $f(x) = \int (x^2 - \sin x + C_1) dx = \frac{1}{3}x^3 + \cos x + C_1x + C_2$. Examining the choices, only A has the correct number of terms, coefficient on x^3 , and sign on $\cos x$.

47. B. I think it's simplest just to look at the choices and ask yourself what their derivatives look like. The derivative of A is just $3 \sec^2 x$, no constant term. B, obviously, works. The derivative of C is $3 \sec x \tan x + 2$. And the derivatives of both D and E use the chain rule and get much more complicated than the given function.

48. C. This is very like #45. Separate the fractions — I think of this as “un-adding” them. Then

$$\begin{aligned} \text{integrate with the power rule. } \int \frac{x^3 + 5}{x^2} dx &= \int \left(\frac{x^3}{x^2} + \frac{5}{x^2} \right) dx = \int (x + 5x^{-2}) dx \\ &= \frac{1}{2}x^2 - 5x^{-1} + C. \end{aligned}$$

49. A. Square it out, then use the power rule.

$$\int (\sqrt{x} + 5)^2 dx = \int (x + 10\sqrt{x} + 25) dx = \frac{1}{2}x^2 + 10 \cdot \frac{2}{3}x^{3/2} + 25x + C$$

50. A. Power rule. $\frac{2x^{-2}}{-2} + C = -x^{-2} + C = \frac{-1}{x^2} + C$

51. C. $\int \cos(3x) dx = \frac{1}{3} \sin(3x) + C$. You could use u -substitution, but can't you see where the fraction comes from without writing that out?

52. B. No way to take the antiderivative here without multiplying out the binomial.

$$\int (x^3 + 1)^2 dx = \int (x^6 + 2x^3 + 1) dx = \frac{1}{7}x^7 + \frac{2x^4}{4} + x + C$$

53. B. Like 45 and 48, separate the fractions. It's a useful technique.

$$\begin{aligned} \int \frac{e^{-x} + 4}{e^x} dx &= \int \left(\frac{e^{-x}}{e^x} + \frac{4}{e^x} \right) dx = \int (e^{-2x} + 4e^{-x}) dx = -\frac{1}{2}e^{-2x} - 4e^{-x} + C \\ &= -\frac{1}{2}(e^{-2x} + 8e^{-x}) + C \end{aligned}$$

54. E. And another one. You do actually need to be good at e^x sorts of problems. They should not require lots of effort from you once you know what you're doing.

$$\int \frac{e^{2x} - e^{3x}}{e^x} dx = \int \left(\frac{e^{2x}}{e^x} - \frac{e^{3x}}{e^x} \right) dx = \int (e^x - e^{2x}) dx = e^x - \frac{1}{2}e^{2x} + C$$

55. B. I did #51 without u -substitution, so I will do this one with that. Let $u = 2x$, the inside function. Then $du = 2dx$. We multiply by 2 and $\frac{1}{2}$ to get the coefficient for dx in the integral. If $x = -\frac{\pi}{6}$ gives $u = -\frac{\pi}{3}$, and if $x = \frac{\pi}{6}$ gives $u = \frac{\pi}{3}$.

$$\begin{aligned} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin(2x) dx &= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin(2x) \cdot 2 dx = \frac{1}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin(u) du = \frac{1}{2} [-\cos(u)]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \frac{1}{2} \left(\cos\left(\frac{\pi}{3}\right) - \cos\left(-\frac{\pi}{3}\right) \right) = -\frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \right) = 0 \end{aligned}$$

If you're clever, you can get that this is 0 without all that work, by considering the graph of sine with those limits of integration.

56. C. The substitution required, $\sqrt{x} = \sin y$, has to be differentiated to get the substitution for dx :

$\frac{1}{2}x^{-\frac{1}{2}} dx = \cos y dy$, and this is easiest to get into the integral by solving it for dx , much like we would usually have an expression for du . That will give $dx = 2\sqrt{x} \cos y dy$. It's weird to have x on both sides there, but remember that $\sqrt{x} = \sin y$. Then $dx = 2 \sin y \cos y dy$. If we just make that substitution, we'll need to replace the limits of integration that are currently values of x for some that are values of y . When $x = \frac{1}{2}$, $\sqrt{x} = \sin y$ becomes $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} = \sin y$, and therefore $y = \frac{\pi}{4}$. When $x = 0$, $\sqrt{x} = \sin y$ gives $0 = \sin y$, so the lower limit will be $y = 0$. Making the

substitutions: $\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{\sqrt{1-x}} dx = \int_0^{\frac{\pi}{4}} \frac{\sin y}{\sqrt{1-\sin^2 y}} \cdot 2 \sin y \cos y dy$. Still kind of a mess, but

there's a trig identity in that denominator; since $\sin^2 y + \cos^2 y = 1$, $1 - \sin^2 y = \cos^2 y$. Then

$$\int_0^{\frac{\pi}{4}} \frac{\sin y}{\sqrt{1-\sin^2 y}} \cdot 2 \sin y \cos y dy = \int_0^{\frac{\pi}{4}} \frac{2 \sin^2 y \cos y}{\sqrt{\cos^2 y}} dy = \int_0^{\frac{\pi}{4}} \frac{2 \sin^2 y \cos y}{\cos y} dy$$

$$= \int_0^{\frac{\pi}{4}} 2 \sin^2 y dy. \text{ This seems a little outdated now; it's a great problem, but I doubt you are going}$$

to need a trig identity.

57. B. Clearly the argument of f must be our u , so $u = x^2 - 1$ and $du = 2x dx$. Then the lower limit of $x = 1$ becomes $u = 1^2 - 1 = 0$, and the upper limit of $x = 3$ will be $u = 3^2 - 1 = 8$. To make

the du fit the integrand, we need a 2, so $\int_1^3 x f(x^2 - 1) dx = \frac{1}{2} \int_1^3 2x f(x^2 - 1) dx$

$$= \frac{1}{2} \int_0^8 f(u) du = \frac{1}{2} \cdot 6 = 3.$$

58. D. The inside here becomes $u = x^3 - 1$, and then $du = 3x^2 dx$. Making the 3 into a coefficient requires also multiplying by $\frac{1}{3}$, so $\int x^2(x^3 - 2)^{10} dx = \frac{1}{3} \int 3x^2(x^3 - 2)^{10} dx = \frac{1}{3} \int u^{10} du$.

Antidifferentiating, we get $\frac{1}{3} \cdot \frac{u^{11}}{11} + C = \frac{(x^3 + 1)^{11}}{33} + C$.

59. B. Once again we identify the inside as $u = x^3$ and differentiate to get that $du = 3x^2 dx$.

Multiplying and dividing by 3 rewrites the integral as $\int x^2 \cos(x^3) dx = \frac{1}{3} \int 3x^2 \cos(x^3) dx$, and

then substituting, we have $\frac{1}{3} \int \cos u du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(x^3) + C$.

60. C. This is again u -substitution. Let $u = \frac{1}{2}x$, so that $du = \frac{1}{2}dx$. Then the limits of integration

become $u = \frac{1}{2} \cdot 6 = 3$ and $u = \frac{1}{2} \cdot 10 = 5$. Our integral becomes

$$\int_6^{10} f\left(\frac{1}{2}x\right) dx = 2 \int_3^5 f\left(\frac{1}{2}x\right) \cdot \frac{1}{2} dx = 2 \int_3^5 f(u) du. \text{ Since we know that } f(x) \text{ is the}$$

derivative of $g(x)$, it must be true that g is the antiderivative of f . Then

$$2 \int_3^5 f(u) du = 2 [g(x)]_3^5 = 2(g(5) - g(3)).$$

61. A. This one is u -substitution, with $u = \sqrt{x}$ and $du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$. Then

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx = 2 \int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

62. A. If $u = x^2 + 1$, then $du = 2x dx$. That still means we need to deal with the $x^2 - 1$ in the numerator. That's $u - 2$. The lower limit becomes $0^2 + 1 = 1$, and the upper limit will now be

$$2^2 + 1 = 5. \text{ So } \int_0^2 \frac{2x(x^2 - 1)}{x^2 + 1} dx = \int_1^5 \frac{u - 2}{u} du.$$

63. C. Tricky. Separate the integrand into two fractions and make them nicer first.

$$\int \frac{x + 1}{3x^2} dx = \int \left(\frac{x}{3x^2} + \frac{1}{3x^2} \right) dx = \int \left(\frac{1}{3x} + \frac{1}{3x^2} \right) dx$$

This will be even less annoying if we factor out the 3.

$$\frac{1}{3} \int \left(\frac{1}{x} + \frac{1}{x^2} \right) dx = \frac{1}{3} \int \left(\frac{1}{x} + x^{-2} \right) dx. \text{ And now it's manageable.}$$

$$\frac{1}{3} \left(\ln |x| + \frac{x^{-1}}{-1} \right) + C = \frac{1}{3} \ln |x| - \frac{1}{3x} + C$$

64. D. Let $u = x^4 + 1$, and then $du = 4x^3 dx$.

$$\int \frac{x^3}{x^4 + 1} dx = \frac{1}{4} \int \frac{4x^3}{x^4 + 1} dx = \frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |x^4 + 1| + C$$

The absolute value is optional because $x^4 + 1 > 0$ for all values of x .

65. E. Let $u = e^x + 4$; then $du = e^x dx$. $\int \frac{e^x}{e^x + 4} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |e^x + 4| + C$

Once again, the absolute value is not required, because the argument is positive.

66. A. This one is long division, and I don't have the patience to do that in LaTeX at the moment, so you get handwriting. We can tell it ought to be divided because the degree of the numerator is greater than the degree of the denominator (equal degrees would also call for this). Once that's done,

we rewrite the integral. $\int_1^2 \frac{x^2 - x - 5}{x + 2} dx = \int_1^2 \left(x - 3 + \frac{1}{x + 2} \right) dx$

$$\begin{array}{r} x - 3 + \frac{1}{x + 2} \\ x + 2 \overline{) x^2 - x - 5} \\ \underline{-(x^2 + 2x)} \\ -3x - 5 \\ \underline{-(-3x - 6)} \\ 1 \end{array}$$

$$= \left[\frac{1}{2}x^2 - 3x + \ln |x + 2| \right]_1^2 = \left(\frac{1}{2} \cdot 4 - 3 \cdot 2 + \ln 4 \right) - \left(\frac{1}{2} \cdot 1 - 3 \cdot 1 + \ln 3 \right)$$

$$= 2 - 6 + \ln 4 - \frac{1}{2} + 3 - \ln 3 = -\frac{3}{2} + \ln 4 - \ln 3. \text{ It's a law of logarithms that tells us}$$

$\ln 4 - \ln 3 = \ln \frac{4}{3}$, but even without that, you'd be able to identify the correct choice.

67. D. This one is completing the square. I'll do that, but first, look at the choices. If one were to go by appearances, they'd probably reject A immediately, and narrow down to a choice with both a coefficient of 8 and a plus in the numerator of the fraction. Hmm. The formula one might choose to learn for arcsine integrals says that $\int \frac{1}{\sqrt{a^2 - u^2}} du = \arcsin \frac{u}{a} + C$. Working on just the integrand, $12 - x^2 - 4x = 12 - (x^2 + 4x) = 12 + 4 - (x^2 + 4x + 4) = 16 - (x + 2)^2$. Notice that the "+ 4" in the parentheses is really adding a negative because of the minus on those parentheses. That's why it took a "+ 4" with the 12 to balance it. Now the original integral becomes $\int \frac{8}{\sqrt{12 - x^2 - 4x}} dx = 8 \int \frac{1}{\sqrt{16 - (x + 2)^2}} dx$. Since choosing $a = 4$ and $u = x + 2$ gives $du = dx$ exactly matches the formula $\int \frac{1}{\sqrt{a^2 - u^2}} du = \arcsin \frac{u}{a} + C$, the antiderivative is $8 \arcsin \left(\frac{x + 2}{4} \right) + C$.

68. B. The degrees of numerator and denominator match, so long division works here. Again to the pen. The integral becomes $\int \frac{x - 1}{x + 1} dx = \int \left(1 - \frac{2}{x + 1} \right) dx = x - 2 \ln |x + 1| + C$.

Handwritten long division of $\frac{x-1}{x+1}$. The divisor is $x+1$ and the dividend is $x-1$. The quotient is 1, and the remainder is -2 . The result is $1 - \frac{2}{x+1}$.

69. C. Did you recognize the long division this time? The numerator degree is larger. So $\int_3^5 \frac{2x^2 + x + 4}{x - 1} dx = \int_3^5 \left(2x + 3 + \frac{7}{x - 1} \right) dx$
 $= \left[x^2 + 3x + 7 \ln |x - 1| \right]_3^5 = (25 + 15 + 7 \ln 4) - (9 + 9 + 7 \ln 2)$
 $= 22 + 7 \ln 4 - 7 \ln 2 = 22 + 7(\ln 4 - \ln 2)$.

Handwritten long division of $\frac{2x^2 + x + 4}{x - 1}$. The divisor is $x - 1$ and the dividend is $2x^2 + x + 4$. The quotient is $2x + 3$, and the remainder is 7 . The result is $2x + 3 + \frac{7}{x - 1}$.

70. D. I hope it's clear to you that I is false. There's just no rule that works that way. You could prove they're different by evaluating I and the original integral and showing that the resulting values are different. Long division once more time gives the integrand for II, and that one is equivalent to the original. Using $u = x + 2$ gives $du = dx$; the lower limit of $x = 0$ results in $u = 2$ and the upper limit of $x = 5$ gives $u = 7$. That makes III correct, as well.

Handwritten long division of $\frac{3 + x + 2}{x + 2}$. The divisor is $x + 2$ and the dividend is $3 + x + 2$. The quotient is 1, and the remainder is 5. The result is $1 + \frac{5}{x + 2}$.

71. B. The easiest thing to try is to distribute.

$$\int \sqrt{x} \left(9x^3 - 2\sqrt{x} + \frac{8}{x} \right) dx = \int x^{1/2} (9x^3 - 2x^{1/2} + 8x^{-1}) dx = \int (9x^{7/2} - 2x^1 + 8x^{-1/2}) dx.$$

Winner. No need to check the others, really, but if you want to know why the others aren't true...

Choice A is just silly; that's not how integrals work. For C and D, if $u = \sqrt{x}$,

$$du = \frac{1}{2} x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx.$$

Choice C doesn't make the du work, and D brings the variable outside the integral. That's not allowed.