Worked solutions to Unit 5 (Analytical Applications of Differentiation) problems from notes

1. D. The Mean Value Theorem says that if $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a value of c between a and b such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. If you divide both sides of choice D by $(b - a)$, it will look like the right formula. The problem with choice A is that the numerator and denominator have reversed the order of subtraction.

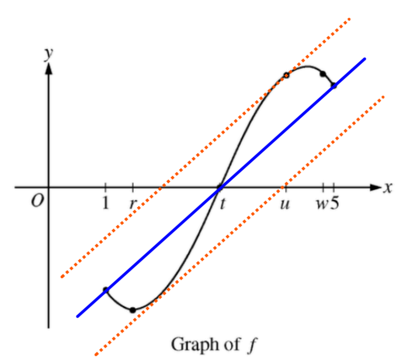
2. (a) This is the MVT again. You can read that in the previous answer. Since f is differentiable, it is also continuous, and we can calculate $\dfrac{f(5) - f(2)}{5 - 2} = \dfrac{2 - 5}{5 - 2} = -1$. The MVT guarantees that there must be a value of c such that $2 < c < 5$ at which $f'(c) = -1$. (And, yes, you can name that theorem by its initials. That's also true for the IVT, EVT, and FTC.)

   (b) First, we have to calculate $g'(x)$, which uses the chain rule. $g'(x) = f'\left(f(x)\right) \cdot f'(x)$. Then $g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$. I don't know either of those two values, but maybe that will still be okay. Checking, $g'(5) = f'(f(5))f'(5) = f'(2)f'(5)$. Sure enough, $g'(2) = g'(5)$. Since f is twice-differentiable, f' is both continuous and differentiable, so g' is also continuous and differentiable. Once again using the Mean Value Theorem, this time on g', we have that there exists a value of k where $2 < k < 5$ and $g''(k) = \dfrac{g'(5) - g'(2)}{5 - 2} = 0$.

   (c) The graph of g will have a point of inflection where g'' changes signs, so first we calculate that derivative using the product rule on g': $g''(x) = f'(f(x)) \cdot f''(x) + f'(x) \cdot f''(f(x))f'(x)$. If $f''(x) = 0$ no matter what its input, then $g''(x) = f'(f(x)) \cdot 0 + f'(x) \cdot 0 = 0$, and g'' will never change signs. It cannot have a point of inflection. [There is an entirely different way to think about this; if $f''(x) = 0$ all the time, f itself must be linear, and therefore g is linear, too. Linear functions have no points of inflection.]

   (d) Parts (a) and (b) were about specific values of a derivative, which is what the MVT addresses. This part is about a y-value of the function itself. That's likely to be the Intermediate Value Theorem. The IVT says that if $f(x)$ is continuous on $[a, b]$ and k is any value between $f(a)$ and $f(b)$, then there exists a value of c in the interval $a < c < b$ at which $f(c) = k$. Since f is continuous, $h(x) = f(x) - x$ is also continuous. The y-values of h at the endpoints of the given interval are $h(2) = f(2) - 2 = 5 - 2 = 3$ and $h(5) = 2 - 5 = -3$. Because $h(2) > 0 > h(5)$, there must be a value of r in the interval $2 < r < 5$ at which $h(r) = 0$.

3. C. This one really invites sketching on the graph. I've drawn in the secant line through those endpoints in blue. There are two different places in the image where the slope of the tangent line matches the slope of that secant line; those are in orange. One of those is at u (and the other has no designated name here).



Graph of f

4. C. Continuity on a closed interval alone is sufficient for the Intermediate Value Theorem and the Extreme Value Theorem, but not Rolle's or the Mean Value Theorem. Choice A is sort of Intermediate Value, but it would require that we know that $f(a)$ and $f(b)$ are on opposite sides of 0,

and we don't. Choice B doesn't have to be true because of the strict inequality; if f were a constant function, all of those values would be equal. Choice C is the Extreme Value Theorem, so that's a yes. Choice D is Rolle's Theorem, and there's insufficient evidence for that. Choice E is the Mean Value Theorem, similarly too little evidence.

5. B. A and C are true by the Intermediate Value Theorem. D is true by the Mean Value Theorem, since the slope between the points (–2, –5) and (1, 4) is 3. E is true by the Extreme Value Theorem. But the graph of f might just be a straight line connecting the given endpoints, and it's not necessary that its slope be 0 anywhere.

6. C. The MVT is about average slopes, so we're looking for places where the $\frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x}$ from a to 4 is equal to 3. However, the hypotheses of the theorem require that $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). Since $f(x)$ has corners at both $x = 0$ and $x = 2$, any interval that has 4 as its right endpoint can't have a left endpoint smaller than 2, because the differentiability requirement would be violated. Given the choices, that's III only. Checking the slope, $\frac{f(4)-f(2)}{4-2} = \frac{12-6}{2} = 3$, and III works.

7. B. This is very much like #2, just as a multiple-choice question. Since $f(x)$ is twice-differentiable and $g(x)$ is a composite of $f(x)$ with itself, $g(x)$ is also twice-differentiable. So the MVT should apply to $g'(x)$. It will guarantee a value of k (usually called c) where its instantaneous rate of change equals its average rate of change. The instantaneous r. o. c. of $g'(x)$ is $g''(x)$. The average rate of change of $g'(x)$ is $\frac{g'(b)-g'(a)}{b-a}$. Evaluating that numerator is the issue. By the chain rule, $g'(x) = f'\left(f(x)\right)f'(x)$, so $g'(b) = f'\left(f(b)\right)f'(b) = f'(a)f'(b)$, and $g'(a) = f'\left(f(a)\right)f'(a) = f'(b)f'(a)$. Hey, those are the same. Putting it all together, the MVT guarantees a value of k so that $g''(k) = \frac{f'(a)f'(b)-f'(b)f'(a)}{b-a} = 0$.

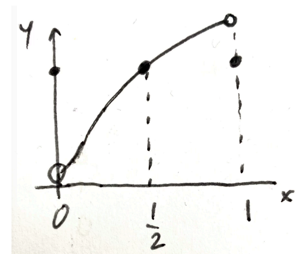
8. C. Rolle's theorem says that if $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there's a c in the open interval where $f'(c) = 0$. Clearly $f(x)$ is both continuous and differentiable because it's a polynomial. And $f(1) = f(0) = 0$. So Rolle's theorem holds. Since $f'(x) = 1 - 3x^2$, the zero is at $x = \sqrt{\frac{1}{3}} \approx 0.577$.

9. C. III is the easy one to check, clearly both continuous and differentiable, and $f(0) = f(2) = 0$, so III works, which eliminates choices A and B. Checking II next: it has a corner at $x = 1$ and isn't differentiable there. That eliminates D and E. Done.

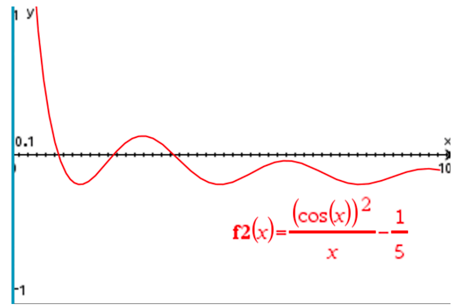
10. E. The MVT only requires that the function be continuous on the closed interval and differentiable on the open interval. I is continuous and differentiable everywhere. That eliminates choices B and C. II will fail to be differentiable at $x = 8$ because of a vertical tangent line, but that isn't on the interval, so II works, too, and we can eliminate A. A graph of III would have corners where

$x^2 - 2x$ changes signs, at 0 and 2, but it is continuous everywhere, and differentiable *between* those values, so it works, too, and the answer is E.

11. D. At first read, it's hard to understand how $f(x)$ could be continuous, strictly increasing, *and* have the endpoint y-values the same as each other. The catch here is that the continuity is only on the *open* interval. The endpoints don't have to be connected. Here's a way that might look. No absolute maximum or minimum is present. The Extreme Value Theorem says those must exist if the function is continuous on a *closed* interval.



12. B. Critical numbers are values of x in the domain of the function where its derivative is either zero or undefined. As we know the derivative already, we can examine its graph. I've set the domain here to $0 \leq x \leq 10$. It took a couple of adjustments to get the vertical interval to show clearly where the graph crosses the axis; I ended up with $-1 \leq y \leq 1$. You can clearly see that this hits the x-axis three times.



13. A. The given information sounds a bit like the Extreme Value Theorem; that guarantees both a maximum and a minimum value of the function if it is continuous on a closed interval. We know that f is defined on the closed interval, so if there *isn't* a maximum, f must not be continuous.

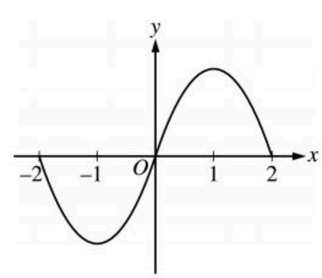
14. C. Looking at the choices, it's probably only necessary to check the signs of $f'(x)$ to the right of $x = 1$ and of $f''(x)$ to the right of e. To the right of $x = 1$, $f'(x) > 0$, which narrows the choices down to C and D. To the right of e, $f''(x) < 0$, which makes it C.

15. (a) Since $g(x) = \int_0^x f(t)dt$, the Fundamental Theorem of Calculus tells us that $g'(x) = f(x)$ and therefore we also know that $g''(x) = f'(x)$. Then $g(-1) = \int_0^{-1} f(t)dt = -\int_{-1}^{0} f(t)dt$ $= -\frac{1}{2} \cdot 1 \cdot 3$, $g'(-1) = f(-1) = 0$, and $g''(-1) = f'(-1) = 3$, which can be found by counting the slope of the given graph.

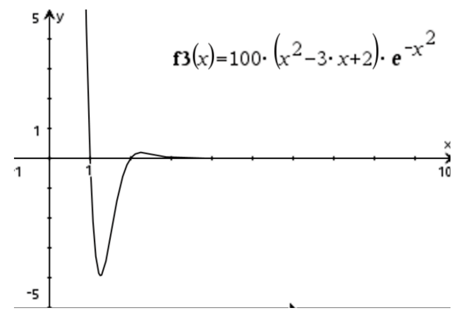
(b) The graph of g will be increasing where $g' = f$ is positive, on $(-1, 1)$. That's an open interval only because the question asks for that.

(c) A graph is concave down when its second derivative is negative; $g'' = f'$ is negative for $0 < x < 2$.

(d) To get a graph, we need some points. We know that $g(-1) = -\frac{3}{2}$ and the symmetry of the graph makes $g(1) = \frac{3}{2}$. The integral function will equal 0 when $x = 0$, and the equal areas above and below the x-axis mean that $g(-2)$ and $g(2)$ are also equal to 0. The graph is shown to the right.



16. B. To check whether $g(x)$ is increasing, we need $g'(x)$. The FTC gives $g'(x) = 100\left(x^2 - 3x + 2\right)e^{-x^2}$. A graph of that shows that $g'(x) < 0$ on $(1, 2)$, so option I is false, but $g'(x) > 0$ on $(2, 3)$, so II is true. To check option III, the calculator can evaluate that integral, $g(3) = \int_1^3 100\left(t^2 - 3t + 2\right)e^{-t^2}dt \approx -1.942$, which is not positive. Only II is true.



17. D. First differentiate, as we're looking for where $f'(x)$ is positive; that requires the product rule: $f'(x) = \sin x \cdot e^{-x} \cdot -1 + e^{-x} \cdot \cos x = e^{-x}(-\sin x + \cos x)$. Setting that equal to zero will let us find the critical numbers, and either $e^{-x} = 0$ or $-\sin x + \cos x = 0$. Since e is positive, e^{-x} is never 0, so it must be true that $\sin x = \cos x$, and $x = \dfrac{\pi}{4}$ is the only solution on the given domain. (There would be another one in quadrant three where both of those functions are negative, but f never gets that far.) Checking the signs of $f'(x)$ to the left and right of $x = \dfrac{\pi}{4}$ gives a positive value only to the left, since $f'(0) = e^0(-\sin 0 + \cos 0) = 1(0 + 1) = 1 > 0$, but $f'\left(\dfrac{\pi}{3}\right) = e^{-\pi/3}\left(-\sin\dfrac{\pi}{3} + \cos\dfrac{\pi}{3}\right) = (\text{positive})\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\right) = \text{positive} \cdot \text{negative} < 0$. That means $f(x)$ is increasing on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$.

18. D. For critical numbers, we look for where $f'(x) = 0$; that's at $x = \dfrac{1}{2}, -\dfrac{1}{3}$, and 1. Because of the second power, though, $f'(x)$ can't change signs at $x = \dfrac{1}{2}$, and II cannot be true. At $x = -\dfrac{1}{3}$, $f'(x)$ changes signs from positive to negative, which gives a maximum at that value, and I is true. At $x = 1$, $f'(x)$changes signs from negative to positive, so there's a minimum here, making III true.

19. (a) We have the formula for $P'(t)$, so what we need is its sign at $t = 9$: $P'(9) \approx -0.646 < 0$, so the amount of pollutant is not increasing at $t = 9$.

    (b) The minimum can occur at a critical point or an endpoint. The only endpoint we have is when $t = 0$ and $P(0) = 50$. For critical points, we look for where the derivative, $P'(t)$, is zero or undefined. That is never undefined for $t \geq 0$, and $P'(t) = 0$ for $t \approx 30.174$. Since there is no right endpoint to compare, we need to consider the signs of P' to see if this is a minimum (or if P might continue to decrease after a momentary stationary point — it really is necessary to check). You can just graph $P'(t)$ on your calculator to be able to assert something about its signs rather than doing algebra. We can just write that $P'(t) < 0$ for $0 \leq t < 30.174$ and $P'(t) > 0$ for $t > 30.174$, so the minimum amount of pollutant is when $t \approx 30.174$.

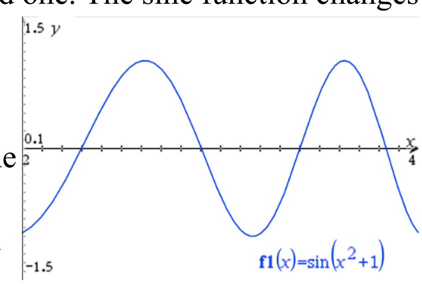
    (c) This requires figuring out how many gallons of pollutant are in the lake at the time we just found. It's an accumulation question, final = initial + $\int_a^b (\text{rate of change})\, dt$.

$P(30.174) = 50 + \int_0^{30.174} P'(t)dt \approx 35.104$ gallons. Since this is less than 40 gallons, the lake will be safe at this time.

(d) For the tangent line, we use the point (0, 50) and the slope then of $P'(0) = 1 - 3e^0 = -2$. That tangent line is $y - 50 = -2(t - 0)$. If we let $y = 40$, the safe amount, we get $40 - 50 = -2t$, and $t = 5$ days.
[I was curious, so I thought it would be interesting to use the integral above to find $P(5)$. $P(5) = 50 + \int_0^5 P'(t)dt \approx 43.8038$. Not safe yet. It really looks to be just over 10 days before it's going to be safe. That's because $P'(t)$ is an increasing function, so $P''(t) > 0$. That makes P concave up; every linear approximation will be an underestimate.]

20. A. Critical numbers are $x = 2$, 3, and 4, but the second power means that the sign of $f'(x)$ cannot change at $x = 3$, and II is out as a possibility, as are answers D and E. I'll check signs in the intervals. The sign of $f'(x)$ changes from positive to negative at $x = 2$, producing a maximum, and making I true. The sign of $f'(x)$ changes from negative to positive at $x = 4$, giving a minimum at $x = 4$, and that makes III false. So it's A.

21. D. While you would have a calculator for this, but you don't need one. The sine function changes signs every time it is equal to zero, and sine is 0 at integer multiples of π. So $f'(x)$ will change signs when $x^2 + 1 = 0, \pi, 2\pi, 3\pi, 4\pi$, and so on. If x is between 2 and 4, then $x^2 + 1$ is between 5 and 17 (go ahead, think that through for a second). The first multiple of π in that interval is 2π, then 3π, 4π, and the last one is 5π. Counting up, that gives me four times. Here is a graph of $f'(x)$ to convince you.



22. C. First find the critical numbers: $f'(x) = 6x^2 - 6x - 12 = 0$ gives $6(x^2 - x - 2) = 6(x - 2)(x + 1) = 0$, so $x = 2, -1$.
$f'(x)$ changes signs from negative to positive at $x = 2$, so that's where the minimum is.

23. B. $f'(x)$ has a minimum at $x = 1$, but f' isn't the same as f. Since $f'(1)$ is negative, $f(x)$ must be decreasing there. Choice A is true because $f'(x) > 0$ (above the x-axis) on $2 < x < 3$. Choice C is true because f' changes signs from positive to negative at $x = 0$. Choice D is true because, despite the corner in this graph, this *is* $f'(x)$. We can see that it exists at $x = 3$, so f itself must be differentiable at that location. Finally, E is true because $f'(x)$ is decreasing on $-2 < x < 1$, so $f''(x) < 0$ there.

24. C. There's a maximum at wherever that zero of $f'(x)$ is on the left-hand side of the graph because $f'(x)$ changes signs from positive to negative there. If you set $1 - 2e^{x+1} = 0$, you get that $x = \ln\left(\frac{1}{2}\right) - 1$, which is the same thing as $-1 - \ln 2$ (kind of sneaky, though). So I is true.
Because $f'(x)$ has a local minimum at $x = 1$, its derivative, $f''(x)$, must change signs at that location,

so $f(x)$ has a point of inflection there. Therefore II is true. However, $f'(x) < 0$ on $(1, 3)$, and III isn't true.

25. D. $g'(x) = 2x - f'(x)$, so $g'(x) = 0$ when $2x = f'(x)$. If you graph $y = 2x$ on that given diagram with $f'(x)$, you'll see that they intersect when $x = 0$ and $x = 2$, and those are the critical numbers. To the left of $x = 0$, $g'(x) > 0$ (because $2x > f'(x)$ in the graph you made). Between $x = 0$ and $x = 2$, $g'(x) < 0$, and to the right of $x = 2$, $g'(x) > 0$. That means we have a maximum at $x = 0$ and a minimum at $x = 2$.

26. E. Absolute extrema can happen at critical numbers and endpoints. The only critical number is at $x = 1$, where $f'(x) = 0$. At that location, though, $f'(x)$ changes signs from positive to negative, producing a maximum value. How can we decide whether the maximum is at the left or right endpoint? Compare the accumulated change in f. The area between the x-axis and the curve is how much the y-values change by (the definite integral of a rate of change of y gives the amount of change of y). There is a vertical scale, and we can clearly see that the area enclosed above the x-axis on the interval $[0, 1]$ is less than the area enclosed below it on $[1, 2]$. The value of y has decreased more than it increased, and the minimum must be at the right endpoint.

27. B. While one could analyze the derivatives here, I'd rather just type some integrals at this point. The number of people in the museum at time x can be calculated as the integral of the net rate of change (in minus out) over the time period from 0 to x: $P(x) = \int_0^x \big(f(t) - g(t)\big)\,dt$. Since one of the listed values is the answer, I'm just going to try them all. (This is more of a testing strategy than calculus analysis, although the integral requires a good understanding of accumulation. Its worth pointing out that you can use the multiple-choice format to your advantage at times.) The largest value happens at $t = 7.888$.

Define $fp(t)=380\cdot(1.02)^t$ — *Done*

Define $gp(t)=240+240\cdot\sin\left(\frac{\pi\cdot(t-4)}{12}\right)$ — *Done*

$\int_0^1 (fp(t)-gp(t))\,dt$ — 333.649

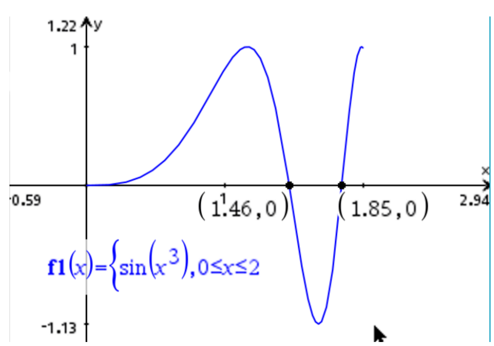
$\int_0^{7.888} (fp(t)-gp(t))\,dt$ — 1374.18

$\int_0^{10.974} (fp(t)-gp(t))\,dt$ — 1334.56

$\int_0^{9.446} (fp(t)-gp(t))\,dt$ — 1354.23

$\int_0^{11} (fp(t)-gp(t))\,dt$ — 1334.57

28. C. I'll start with a graph on the interval. Because we're looking for an extreme value, I'll go ahead and mark the coordinates of locations where f' changes signs. The critical point at $x \approx 1.46$ is going to be a local maximum, but since $f'(x) > 0$ to the right of $x = 1.85$, the endpoint at $x = 2$ is also a possibility. Since the area beneath the x-axis on $[1.46, 1.85]$ is more than the area above on $[1.85, 2]$, $f(2) < f(1.46)$, and the maximum is at $x \approx 1.46$. You could also calculate integrals to get numbers to compare.



29. (a) Using the two times closest to $t = 10$, we get $A'(10) \approx \dfrac{A(15) - A(5)}{15 - 5} = \dfrac{25 - 18}{10} = \dfrac{7}{10}$ gallons per hour.

(b) The theorem that may guarantee any specific value of the derivative is the Mean Value Theorem. Since A is twice differentiable, it is also continuous, and the MVT will apply. That means there is a value of c between $t = 0$ and $t = 30$ (the bounds specified in this part) at which $A'(c) = \frac{A(30) - A(0)}{30 - 0} = \frac{16 - 10}{30} = \frac{6}{30} = \frac{1}{5}$. Therefore there *is* such a time t.

(c) The absolute maximum will happen at a critical point or an endpoint. Critical points are found where the derivative of G is either zero or undefined, so we start with a derivative. $G'(t) = 5 - \frac{2}{3} \cdot \frac{3}{2}(t+9)^{\frac{1}{2}} \cdot 1 = 5 - \sqrt{t+9}$. That is never undefined on the domain [0, 35], and $G'(t) = 5 - \sqrt{t+9} = 0$ when $\sqrt{t+9} = 5$, and $t = 16$. That's the only critical number of G. I initially started to do the Candidates Test for this, where you substitute the endpoints and critical numbers into the function to compare the output values, but the function is a little uglier than I would like without a calculator to back me up, so I will instead make an argument based on the fact that there is just one critical number. The sign of $G'(t)$ changes from positive to negative at $t = 16$, and G is continuous with only one critical number on the given interval. Therefore the local maximum at $t = 16$ is also the absolute maximum. Note that you have to be really careful with this sort of argument; if I didn't have to *type* all of the calculations with those fractions, I would probably just compute those three y-values!

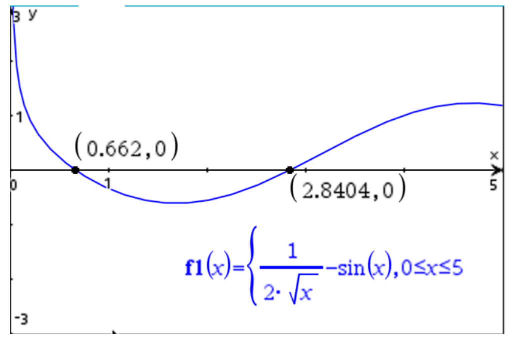
(d) Linear approximations are overestimates if the function they approximate is concave down, and underestimates when the function being approximated is concave up. To decide that, we need the second derivative of G: $G''(t) = \frac{d}{dt}\left(5 - \sqrt{t+9}\right) = -\frac{1}{2}(t+9)^{-\frac{1}{2}}$, and $G'' < 0$ for all values of $t \in [7, 8]$, so the approximation must be an overestimate because G is concave down there.

30. B. Checking critical points and endpoints: $f'(x) = 6x - 3x^2 = 0$ gives $3x(2 - x) = 0$, so the critical numbers are $x = 0$ and $x = 2$. Of those, only $x = 2$ is on the domain. The candidates test says that we can just calculate the y-values at both endpoints and any critical points between them, then pick the smallest number as the minimum. I get $f(1) = 3 \cdot 1^2 - 1^3 = 2$, $f(2) = 3 \cdot 2^2 - 2^3 = 4$, and $f\left(\frac{5}{2}\right) = 3\left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^3 = \frac{75}{4} - \frac{125}{8} = \frac{150}{8} - \frac{125}{8} = \frac{25}{8}$. That last fraction is a bit larger than 3, so the minimum value is $f(1) = 2$.

31. (a) This is an accumulation question, final = initial + $\int_a^b \left(\text{rate of change}\right) dt$. The final distance is $35 + \int_0^5 f(t)dt \approx 26.495$ meters. Notice that the integrand is f, not f'. That's because $f(t)$ is already the rate of change of the distance.

(b) Since f gives the rate of change of the distance between the road and the water at a time t, the derivative of f gives the instantaneous rate of change of that rate of change, which is a little hard to parse. Since the value of $f'(4)$ is positive, something is increasing, and it's important to get that word in your answer. The value of t gives the number of hours after the storm began. Putting all of that together $f'(4) = 1.007$ means that the rate of change of the distance between the road and the edge of the water was increasing at 1.007 meters per hour per hour four hours after the storm began.

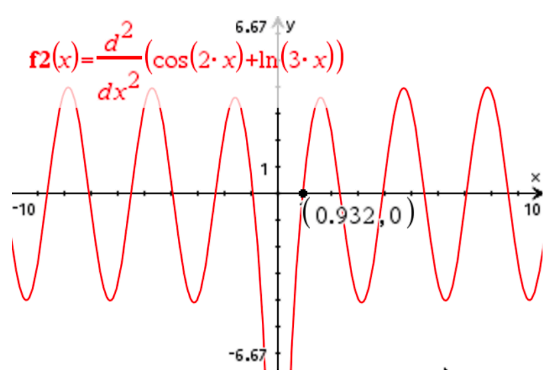
(c) "Decreasing most rapidly" means we are looking for when $f(x)$ has its most negative value. That's an absolute minimum. While looking at a graph of f will give you a numerical answer pretty quickly, that won't earn any marks here; you've got to have calculus to justify any answer. So we check critical points and endpoints once again. Since we have a calculator and knowing the sign changes is useful, graphing is a good way to solve $f'(x) = 0$. The critical numbers are $x \approx 0.662$ and $x \approx 2.840$. We also have the formula for $f(x)$, so it's easy to calculate the distances at the endpoints and both critical points: $f(0) = -2$, $f(0.662) \approx -1.398$, $f(2.840) \approx -2.270$, and $f(5) \approx -0.480$. Therefore the distance is decreasing most rapidly at $t \approx 2.840$ hours after the beginning of the storm.



(d) The amount of sand restored from time 0 to x is $\int_0^x g(p)dp$, and the amount lost during the storm is $\int_0^5 f(t)dt$. However, those have different signs, so the equation that has to be solved must change one of the signs: $\int_0^x g(p)dp = -\int_0^5 f(t)dt$.

32. C. Concavity can be determined with the sign of the second derivative:
$$f''(x) = \frac{e^{-x}(2x-5) - (x^2-5x-5)e^{-x}(-1)}{(e^{-x})^2} = \frac{e^{-x}(x^2-3x-10)}{(e^{-x})^2} = \frac{x^2-3x-10}{e^{-x}}.$$
Then $f''(x) = 0$ when its numerator is 0, so $x^2 - 3x - 10 = (x-5)(x+2) = 0$, and $x = 5$ or -2. Checking signs of x to the left of -2, to the right of 5, and between the two, $f''(x) < 0$ between -2 and 5. That's C.

33. B. The graph of $f(x)$ changes concavity where f'' changes signs. The calculator can graph that for you. As you can see in the image here, the smallest one of the choices that's a place where $f''(x)$ changes signs is $x \approx 0.932$.



34. C. A point of inflection needs a change in signs of f'', so first we calculate $f''(x) = 3x^2 - 4$. Setting that equal to zero gives $x^2 = \frac{4}{3}$ and $x = \pm\frac{2}{\sqrt{3}}$.

35. B. We have two facts here. First, the point $(1, -6)$ is on the curve; secondly, y'' must be zero when $x = 1$. The second derivative first: $y' = 3x^2 + 2ax + b$, and $y'' = 6x + 2a$. Then $y''(1) = 6 \cdot 1 + 2a = 6 + 2a = 0$, and $a = -3$. Using that with the original equation, $y(1) = 1^3 - 3 \cdot 1^2 + b \cdot 1 - 4 = -6$, which gives $-6 + b = -6$, and $b = 0$.

36. (a) On the interval $1.7 < x < 1.9$, the given graph of f' is decreasing, so $f''(x) < 0$. That means $f(x)$ is concave down there.

(b) The critical numbers, where $f'(x) = 0$, are at $x \approx 1.772$ and at $x \approx 2.507$. (That's from a graph.) We know that final = initial + $\int_a^b \left(\text{rate of change}\right) dx$, which here looks like $f(x) = f(0) + \int_0^x f'(t)dt$. Checking the values of the function at the endpoints and critical points gives $f(0) = 5, f(1.772) \approx 5.679, f(2.507) \approx 5.4078$, and $f(3) \approx 5.57893$. Therefore the absolute maximum occurs at $x \approx 1.772$.

(c) The tangent line needs a point and a slope. Using the formula from part (b), $f(2) = f(0) + \int_0^2 f'(t)dt \approx 5.623$. The original formula for f' given in the problem tells us that $f'(2) = e^{(2/4)} \sin(2^2) \approx -0.459$. The tangent line is $y - 5.623 = -0.459(x - 2)$.

37. (a) Differentiating both sides with respect to x gives $2x + 8y\frac{dy}{dx} = 3x\frac{dy}{dx} + y \cdot 3$. The right side needs the product rule. Rearranging that to isolate the derivative, $8y\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2x$. Then $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) This can be done in a couple of different orders. I will use the derivative first. If the tangent line is horizontal, then $\frac{dy}{dx} = 0 = \frac{3y - 2x}{8y - 3x}$, and $3y - 2x = 0$. That means $3y = 2x$ and $y = \frac{2}{3}x$. If $x = 3$, then $y = 2$ will give a horizontal tangent line — but only if the point (3, 2) is on the curve, so we check that: $3^2 + 4 \cdot 2^2 = 25$, and $7 + 3(3)(2) = 7 + 18 = 25$. Therefore there is a point on the curve with $x = 3$ where the tangent line is horizontal, and $y = 2$ there.

(c) Differentiating, $\frac{d}{dx}\left(\frac{3y - 2x}{8y - 3x}\right) = \frac{(8y - 3x)\left(3\frac{dy}{dx} - 2\right) - (3y - 2x)\left(8\frac{dy}{dx} - 3\right)}{(8y - 3x)^2}$. We already know that $\frac{dy}{dx} = 0$ at P, so

$$\left.\frac{d^2y}{dx^2}\right|_{(3,2)} = \frac{(8 \cdot 2 - 3 \cdot 3)(3 \cdot 0 - 2) - (3 \cdot 2 - 2 \cdot 3)(8 \cdot 0 - 3)}{(8 \cdot 2 - 3 \cdot 3)^2}$$

$= \frac{7 \cdot -2 - 0 \cdot -3}{7^2} = -\frac{14}{49}$. Since at P, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, the point is a local maximum. This is the second derivative test in action, and you can envision the concave down curve with a slope of 0 to see why it makes sense.

38. D. In order for there to be a maximum, $f'(2) = 0$ isn't enough by itself. However, if there is only the one critical point and $f(x)$ is twice-differentiable, then $f'(2) = 0$ must also be true for statements II and III. The additional information in II means that f is concave down at the critical point, which means that f must be a maximum by the second derivative test. Also, since absolute extrema can only occur at critical points and endpoints, that inequality says that $f(2)$ is the largest

of the y-values at those points, and it must be the location of the maximum. Therefore II and III are both true.

39. B. Checking one at a time, at $x = 1, f'(1) = 0$ and $f''(1) < 0$. That makes a local maximum, so I is false and II is true. There's no need to check III, as there is no choice with both II and III, but the positive second derivative at the critical number at $x = 4$ gives a minimum rather than a maximum.

40. A. The first thing to check is whether this point is a critical point; $\left.\frac{dy}{dx}\right|_{(1,2)} = 5 \cdot 1 \cdot 2 - 1^2 - 2^2 - 5 = 0$, so it is. For an implicit derivative like this (depending on both x and y), we can differentiate again to check the concavity at the critical point.
$\frac{d}{dx}\left(5xy - x^2 - y^2 - 5\right) = 5x\frac{dy}{dx} + y \cdot 5 - 2x - 2y\frac{dy}{dx}$, and
$\left.\frac{d^2y}{dx^2}\right|_{(1,2)} = 5 \cdot 1 \cdot 0 + 2 \cdot 5 - 2 \cdot 1 - 2 \cdot 3 \cdot 0 = 10 - 2 = 8 > 0$. A positive second derivative at the location of a horizontal tangent means a local minimum.

41. E. Since $f''(x)$ shows slopes of $f'(x), f'(x)$ will be decreasing when $f''(x) \leq 0$, on [4, 6].

42. E. It's often easiest to check where slopes are zero first. The graph of f' has x-intercepts at $x = 1$, $x = 3$, and $x = 6$. Both II and III show slopes of zero at those locations, but I does not, so it's out. Graphs II and III are very similar in shape, only differing in how the derivative fails to exist at $x = 4$, but both of them have positive slopes where $f' > 0$ (on $1 < x < 3$ and $4 < x < 6$) and negative slopes where $f' < 0$ (on $0 < x < 1$, $3 < x < 4$, and $6 < x < 8$), so they both qualify as possible graphs of f.

43. C. All of the choices have $f(0) = 0$, so that's no help. The zeros of f' at $x = 0$, 2, 3, and 4 mean that f must have horizontal tangents at only those locations. Choice A doesn't have a horizontal tangent at $x = 2$, choice B doesn't have one at $x = 4$, choice D doesn't have one at $x = 0$, and choice E has an extra one at $x = 1$. Choice C is the only one that's left. You can also see that the shallow downward slope on $2 < x < 3$ goes with the not-very-big negative values of f' on that interval.

44. D. Checking them in order, at $x = c, f'$ changes signs from positive to negative, so f has a maximum there, and A is false. At $x = b, f'$ has a maximum; since $f'(b) > 0, f$ is increasing at $x = b$, and that can't be the location of a maximum. At $x = a, f'$ changes signs from negative to positive, so f has a local minimum rather than a point of inflection. At $x = b$, the local maximum of f' is exactly what it takes for f to have a point of inflection; since f' changes from increasing to decreasing, f'' must change signs from positive to negative there, and D is the right answer. For completeness, on $c < x < d, f'$ slopes down, so $f'' < 0$, and f itself is concave down.

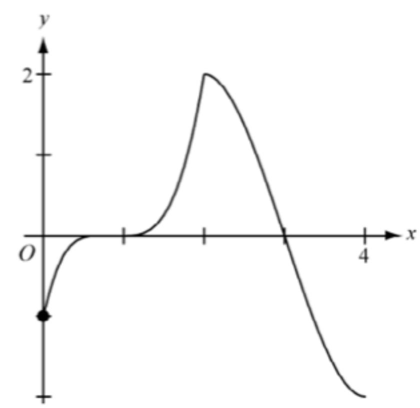
45. A. Knowing that $f(x)$ is concave up means that f' is increasing on that given interval. Since the slope of the given tangent line means $f'(1) = 3$, then $f'(1) < 3$ to the left of 1.

46. E. The derivative of graph I is not present in the diagram, since neither of the other two is equal to zero at the location where I has a horizontal tangent. That makes I the graph of f''. Now f'' has two zeros. At the one on the right, only graph II has a horizontal tangent, so II is f'. That makes III the graph of f.

47. C. The signs of f'' are positive, then negative, then positive. That means the graph of f must be concave up, then down, then up — and that's all. Looking at the choices, A shows concave up, then down. B shows down/up/down. D shows up/down/up/down/up/down. C is the only one that goes concave up/down/up.

48. (a) Relative extrema only happen where at critical points, and those are at $x = 1$ and $x = 2$. At $x = 1, f'$ does not change signs, so that is neither. At $x = 2, f'$ changes signs from positive to negative, producing a local maximum.

(b) The graph of f will start with the four given points: $(0, -1)$, $(1, 0)$, $(2, 2)$, and $(3, 0)$. Since $f'(1) = 0$, f will have a horizontal tangent at (1, 0). We already know that $f(2)$ is a maximum, and the nonexistent derivative there means a corner. It's not a cusp because the second derivative changes signs at $x = 2$, and it's not a vertical tangent line because the location is a maximum. Then getting the concavity to be down on $0 < x < 1$, up on $1 < x < 2$, down on $2 < x < 3$, and up on $3 < x < 4$ gives the graph you see here.



(c) Since $g(x) = \int_1^x f(t)dt$, $g'(x) = f(x)$. The graph of $g(x)$ will have a relative maximum where $g'(x) = f(x)$ changes signs from positive to negative, at $x = 3$, and g will have a relative minimum where $g' = f$ changes signs from negative to positive, at $x = 1$.

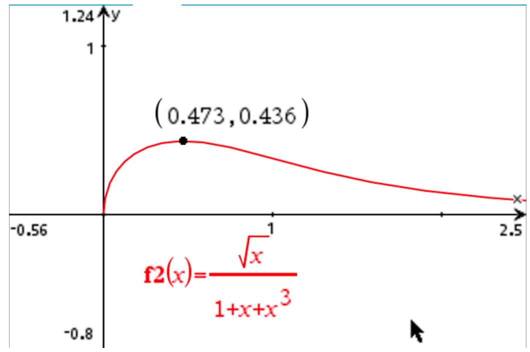
(d) The graph of g will have a point of inflection when $g' = f$ has a local maximum or minimum, at $x = 2$.

49. E. The *derivative, $f'(x)$,* looks like A. Since that's always greater than or equal to zero, f must be increasing. Since $f'(2)$ exists (and is zero), the answer must be E rather than D.

50. D. $f(-2) = 0$. Because f is decreasing, $f'(-2)$ is negative. Because $f(x)$ is concave up, $f''(-2)$ is positive. Therefore $f'(x) < f(x) < f''(x)$ at $x = -2$.

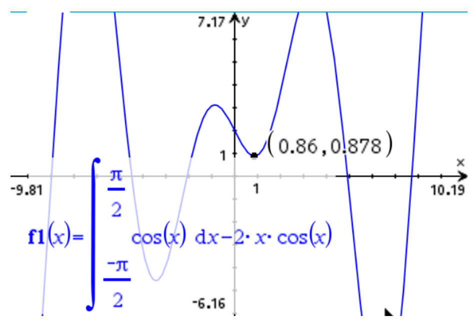
51. B. Positive first derivative means the y-values are increasing as the x-values increase, which eliminates C, D, and E. Negative second derivative means that the slope is getting smaller as x increases. A has it getting larger, because the change in x from one row to the next is the same, but the change in y is getting larger. B has it getting smaller.

52. B. Since you have a calculator for this question, a graph of $f'(x)$ is the best way to go. The graph of f will have a point of inflection where $f''(x)$ changes signs, which is where $f'(x)$ changes from increasing to decreasing or vice versa. So we look for a maximum or minimum on the graph of $f'(x)$. That's at $x \approx 0.473$.



53. C. Since we have the equation of speed, we're looking for the maximum of that given function. It will have a maximum either where $f'(m) = 0$ or at one of the endpoints of the interval.
$f'(m) = \frac{1}{10}(-6m^2 + 18m - 12) = 0$, and factoring gives
$\frac{1}{10} \cdot -6(m^2 - 3m + 2) = -\frac{6}{10}(m-2)(m-1)$. That will be zero when $m = 2$ and $m = 1$.
Checking the values of $f(m)$ at those points and at the endpoints $m = 0$ and $m = 3$ gives $f(0) = 7$, $f(1) = -\frac{5}{10} + 7 = 6.5$, $f(2) = -\frac{4}{10} + 7 = 6.6$, and $f(3) = -\frac{9}{10} + 7 = 6.1$. The maximum of these is 7.

54. B. The x-coordinate of the upper right corner must be $(x, \cos x)$, so the area of the rectangle is $2x \cdot \cos x$. However, that's *not* what we want the minimum value of. The shaded region is $\int_{-\pi/2}^{\pi/2} \cos x \, dx - 2x \cos x$, and graphing that is a really easy way to find the maximum. If this were a free response question, it would be necessary to justify that location with a derivative, but, well, it's not.



55. D. The first task is to decide whether $\frac{dy}{dx}$ exists or not, so we differentiate.
$2x - \left(x\frac{dy}{dx} + y \cdot 1\right) + 2y\frac{dy}{dx} = 0$, and $\frac{dy}{dx} = \frac{-2x + y}{-x + 2y}$. Then $\left.\frac{dy}{dx}\right|_{(2,\,1)} = \frac{-2(2) + 1}{-2 + 2(1)} = \frac{-3}{0}$,
which is surely not defined. That narrows the choices to D and E. A derivative with a zero denominator and a nonzero numerator produces a vertical tangent line.

56. C. While this one can be differentiated implicitly, it's a little simpler to solve for y first. If $x^2y = 4$, then $y = \frac{4}{x^2} = 4x^{-2}$. Then $y' = -8x^{-3}$ and $y'' = 24x^{-4}$. At (2, 1), $y' = -8 \cdot \frac{1}{8} < 0$ and
$y'' = 24 \cdot \frac{1}{16} > 0$.

57. (a) Implicit differentiation: $2y\frac{dy}{dx} = 3x\frac{dy}{dx} + y \cdot 3 - 3$, and then $2y\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3$ and
$\frac{dy}{dx} = \frac{3y - 3}{2y - 3x}$.

(b) $\frac{d}{dx}\left(\frac{3y-3}{2y-3x}\right) = \frac{(2y-3x)\cdot 3\frac{dy}{dx} - (3y-3)\left(2\frac{dy}{dx}-3\right)}{(2y-3x)^2}$. No need to simplify.

(c) The tangent line will be horizontal when the numerator of the derivative is zero; that's when $3y - 3 = 0$, and $y = 1$. If $y = 1$, then using the original equation, $1^2 = 3x - 3x = 0$. But $1 \neq 0$, and there is no point on the curve where the tangent has a slope of 0.

(d) The tangent line will be vertical where the slope has a denominator of 0 and a numerator that isn't 0. If $2y - 3x = 0$, then $2y = 3x$. Substituting that into the original equation, $y^2 = (2y)y - 2y$, which is equivalent to $0 = y^2 - 2y$. That's true for $y = 0$ and $y = 2$. If $y = 0$, then $x = 0$, too, and the first point is (0, 0). When $y = 2$, then $4 = 3x$, and $x = \frac{4}{3}$, so the second point is $\left(\frac{4}{3}, 2\right)$.