Worked solutions to Unit 4 (Contextual Applications of Differentiation) problems from notes

1. E. Since H is the temperature (for heat, maybe?), we know that H' must represent the instantaneous rate of change of the temperature — that's one of the three things you memorized long ago when we first encountered derivatives. The only one of the choices which refers to an instant is E. The word "during" refers to a *duration* (and 10 seconds ago would be the first time I ever recall connecting the roots of those words) rather than an instant. Instantaneous rates of change are *at* some time, not *over* some time interval.

2. D. If $t = f(p)$, then we can think of the derivative $f'(p)$ as $\dfrac{dt}{dp}$ in the same way $y = f(x)$ would make $f'(x)$ the same as $\dfrac{dy}{dx}$. The units of $\dfrac{dt}{dp}$ are $\dfrac{\text{units of } t}{\text{units of } p} = \dfrac{\text{hours}}{\text{psi}}$.

3. Since f gives the rate of change of the distance between the road and the water at a time t, the derivative of f gives the instantaneous rate of change of that rate of change, which is a little hard to parse. Since the value of $f'(4)$ is positive, something is increasing, and it's important to get that word in your answer. The value of t gives the number of hours after the storm began. Putting all of that together $f'(4) = 1.007$ means that the rate of change of the distance between the road and the edge of the water was increasing at 1.007 meters per hour per hour four hours after the storm began.

4. This is one of the "everything" problems from the May 2020 pandemic online exam, hence the seven parts. Barring another emergency shutdown, I doubt you'll ever see more than four parts in a real free response question.
   (a) Average rate of change needs a difference quotient, and by now you probably recall that they're looking to see both the difference and the quotient, but that no simplification is needed. $M'(6) = \dfrac{M(18) - M(0)}{18 - 0} = \dfrac{46 - 40}{18 - 0}$. The units are $\dfrac{\text{units of } M}{\text{units of } t} = \dfrac{\text{coins per day}}{\text{day}}$.
   (b) This is an instantaneous rate of change, and that has to be clear in the explanation: $M'(6)$ is represents the rate at which the rate of change of the number of coins is increasing on day 6. The markscheme specifically says they're looking to see the word "rate" twice here. The day can be referred to with "at" or "on" "day 6" or "the sixth day," and there must be a reference to the number of coins. I recognize that the wording is extremely awkward. They listed several different ways it might be worded, and *all* of them seem awkward to me.
   (c) For a tangent line, we need a point and a slope. For the tangent to $y = C(t)$ at $t = 6$, the point will be $\big(6, C(6)\big)$, or (6, 300). The slope is $C'(6) = M(6) = 54$, from the table. The line is therefore $y - 300 = 54(t - 6)$. Before you ask, they did accept x in place of t there and $C(t)$ in place of y, but the way I typed it uses the preferred variables.
   (d) To find $\displaystyle\int_0^6 M'(3t)dt$, let $u = 3t$, so that $du = 3dt$. When $t = 0$, $u = 0$ for the lower limit; when $t = 6$, $u = 18$ for the upper limit. Multiplying and dividing by 3 to get the expression

for du into the integral and substituting gives us $\frac{1}{3}\int_0^6 M'(3t)\cdot 3dt = \frac{1}{3}\int_0^{18} M'(u)du$. Then the antiderivative is easy. $\frac{1}{3}\left[M(u)\right]_0^{18} = \frac{1}{3}\left(M(18)-M(0)\right) = \frac{1}{3}(46-40) = 2.$

(e) The area of a trapezoid can be calculated with the formula $A = \frac{1}{2}h(b_1+b_2)$, where the height is the horizontal distance between the two vertical bases. You can draw a graph if you like, but I'm not going to. $\int_0^{18} M(t)dt \approx \frac{1}{2}\cdot(6-0)\cdot(40+54)+\frac{1}{2}\cdot(18-6)\cdot(54+46).$

(f) For each additional coin collected, the total weight increases by 3 grams. Therefore the rate at which the weight changes, in total, is three times the rate of change of total number of coins for both classes. That's $3\left(M(7)+B(7)\right)$.

(g) How does this part connect to what came before? I have no idea. I think they were just trying to get a slope field into one of the two long free-response questions that constituted 100% of the online exam that year. The differential equation $\frac{dy}{dt} = 0.3y$ gives slopes that only depend on the value of y; we should see the same slope all the way across each different horizontal set. But we don't; at $y = 1$, for instance, the slopes start at 0 on the y-axis and increase as we go to the right. This can't be the right slope field.
It's also possible to recognize that $\frac{dy}{dt} = 0.3y$ is a special case of $\frac{dy}{dt} = ky$, and the solution of that is one you've seen many times before: $y = Ce^{kt}$. That resulting exponential function must have a horizontal asymptote on the x-axis, but we can see that this slope field has non-zero slopes along the horizontal axis; it cannot be the slope field for our differential equation. That was my first thought, but I decided that noticing slopes only depend on y is both easier and more generally applicable to this sort of question.

5. (a) First the difference quotient: $S''(10) \approx \frac{S(12)-S(8)}{12-8} = \frac{1.7-1.9}{12-8}$. While this does not have to be simplified, it matters that you see it's negative. The interpretation: this is the rate at which the rate of change of the depth of the snow is decreasing at $t = 10$ hours, measured in centimeters per hour per hour.

(b) The rate of change of the snow is given by $S'(t)$, and we have a table full of values of that. We also know that $S(t)$ is twice differentiable, so that $S'(t)$ is both differentiable and continuous. Since $S'(0) = 2.3 > 2 > 1.6 = S'(14)$, so, yes, there is such a value of t. You could also use the values of 3 and 8 for t at the ends. This is an example of the Intermediate Value Theorem, but the name of the theorem isn't required. The connection between the differentiability of S' and its continuity, however, is.

(c) The point here is given as (8, 45), and the slope there will be $S'(8) = 1.9$, so the tangent line we need is $y - 45 = 1.9(t-8)$. Then $S(10) \approx 1.9(10-8)+45$. Given that the graph of S is concave down on its entire domain (there must have been a reason they told us that!), any linear approximation will be an overestimate.

(d) And here's why this is a calculator question. Using the derivative at a point command, $D'(10) \approx 1.791$. At 10 hours, the depth of the snow is decreasing at approximately 1.791 centimeters per hour.

6. D. The functions f and g are themselves rates, so the rate at which the number of people in the building is changing must be $f(t) - g(t)$. We're looking for that rate to be increasing — we need the rate of change of this difference to be positive. That gives $\frac{d}{dt}\left(f(t) - g(t)\right) = f'(t) - g'(t) > 0$.

7. (a) The rate at which the bird's weight changes is $\frac{dB}{dt}$, so we need to compare that value at the two given weights: $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(100 - 40) = 12$, and $\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(100 - 70) = 6$.
Therefore the bird is gaining weight faster when it weighs 40 g.

(b) This involves a bit of implicit differentiation, since B is a function of t.
$\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\frac{1}{5}(100 - B)\right) = \frac{1}{5} \cdot -\frac{dB}{dt}$
However, the question asks for this in terms of B, which means substituting the expression for $\frac{dB}{dt}$: $\frac{d^2B}{dt^2} = \frac{1}{5} \cdot -\left(\frac{1}{5}(100 - B)\right)$. That should tell us something about the shape of the graph of B. The second derivative is generally used to tell about concavity, and $\frac{d^2B}{dt^2} < 0$ for all $B \in [20, 100]$, so the graph of B must be concave down for that range of weights. The graph is concave up for a while, so it can't be the right graph.

(c) The problem helpfully tells us to separate the variables first. $\frac{1}{100 - B}dB = \frac{1}{5}dt$, so $\int \frac{1}{100 - B}dB = \int \frac{1}{5}dt$. That left side can be done with u-substitution, but I'll just point out that the derivative of the denominator will be negative, so I'll need another negative sign to offset that. Feel free to let $u = 100 - B$ if you need more steps.
$-\ln\left|100 - B\right| = \frac{1}{5}t + C$
I'll also notice that the initial condition that $B(0) = 20$ means the argument of the logarithm is positive, so that the absolute value signs aren't needed. Finding the constant of integration looks like $-\ln(100 - 20) = \frac{1}{5} \cdot 0 + C$, and $C = -\ln(80)$. Now we have $-\ln(100 - B) = \frac{1}{5}t - \ln(80)$. Solving will involve dividing the negative over, exponentiating, and isolating the B.
$\ln(100 - B) = -\frac{1}{5}t + \ln(80)$
$100 - B = e^{-\frac{1}{5}t + \ln(80)}$
$B = 100 - e^{-\frac{1}{5}t + \ln(80)}$
That's completely correct as it stands. It will look nicer if I do some work with simplification (and if you solved for B before substituting the initial condition, your answer is likely to look like the one I'm about to end up with).
$B = 100 - e^{-\frac{1}{5}t + \ln(80)} = 100 - e^{-\frac{1}{5}t} \cdot e^{\ln(80)} = 100 - 80e^{-\frac{1}{5}t}$.

8. E. "At rest" means that the velocity is 0, and the velocity is the derivative of position. $v(t) = x'(t) = 6t^2 - 42t + 72 = 0 = 6(t^2 - 7t + 12) = 6(t-3)(t-4)$. That means $t = 3$ and $t = 4$ are the required times.

9. C. The choices make this a calculator question. Acceleration is the derivative of velocity, so use the "Derivative at a Point" command on your calculator: $\left.\frac{d}{dt}\left(v(t)\right)\right|_{t=4} \approx 1.6329$.

10. (a) We have the formula for $P'(t)$, so what we need is its sign at $t = 9$: $P'(9) \approx -0.646 < 0$, so the amount of pollutant is *not* increasing at $t = 9$.

(b) The minimum can occur at a critical point or an endpoint. The only endpoint we have is when $t = 0$ and $P(0) = 50$. For critical points, we look for where the derivative, $P'(t)$, is zero or undefined. That is never undefined for $t \geq 0$, and $P'(t) = 0$ for $t \approx 30.174$. Since there is no right endpoint to compare, we need to consider the signs of P' to see if this is a minimum (or if P might continue to decrease after a momentary stationary point — it really is necessary to check). You can just graph $P'(t)$ on your calculator to be able to assert something about its signs rather than doing algebra. We can just write that $P'(t) < 0$ for $0 \leq t < 30.174$ and $P'(t) > 0$ for $t > 30.174$, so the minimum amount of pollutant is when $t \approx 30.174$.

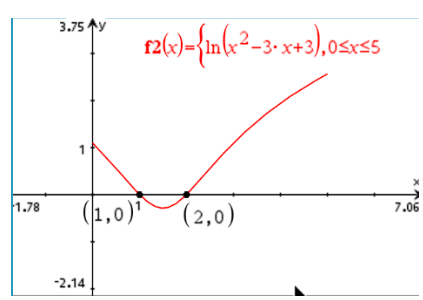
(c) This requires figuring out how many gallons of pollutant are in the lake at the time we just found. It's an accumulation question, final = initial + $\int_a^b \left(\text{rate of change}\right) dt$.

$P(30.174) = 50 + \int_0^{30.174} P'(t)dt \approx 35.104$ gallons. Since this is less than 40 gallons, the lake will be safe at this time.

(d) For the tangent line, we use the point (0, 50) and the slope then of $P'(0) = 1 - 3e^0 = -2$. That tangent line is $y - 50 = -2(t - 0)$. If we let $y = 40$, the safe amount, we get $40 - 50 = -2t$, and $t = 5$ days.

[I was curious, so I thought it would be interesting to use the integral above to find $P(5)$. $P(5) = 50 + \int_0^5 P'(t)dt \approx 43.8038$. Not safe yet. It really looks to be just over 10 days before it's going to be safe. That's because $P'(t)$ is an increasing function, so $P''(t) > 0$. That makes P concave up; every linear approximation will be an underestimate.]

11. (a) Calculator! $a(4) = v'(4) = \left.\frac{d}{dt}\left(\ln(t^2 - 3t + 3)\right)\right|_{t=4} = \frac{5}{7} \approx 0.714$



(b) The particle changes direction when $v(t)$ changes signs. Since we're allowed a calculator here, I'll figure that out with a graph. The particle changes directions at $t = 1$ and $t = 2$. The particle moves to the left when $v(t) < 0$, on $1 < t < 2$.

(c) The final position is the initial position plus the integral of the velocity, so

$$s(2) = s(0) + \int_0^2 v(t)dt = 8 + \int_0^2 \ln(t^2 - 3t + 3)dt \approx 8.369.$$

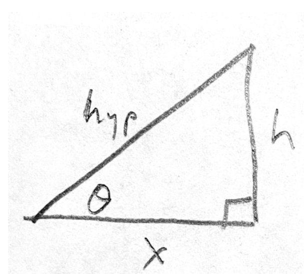
(d) Speed is the absolute value of velocity, and the average value of a function is given by $\frac{1}{b-a}\int_a^b f(x)dx$, so here we have $\frac{1}{2}\int_0^2 \left|v(t)\right| dt \approx 0.371$.

12. C. The particle moves to the left when its velocity is negative, so I'll set $v(t) = 0$.
$v(t) = x'(t) = -30 + 162t - 60t^2 = 0$
$-6(10t^2 - 27t + 5) = -6(5t - 1)(2t - 5) = 0$, so $t = \frac{1}{5}$ or $t = \frac{5}{2}$.
Make a number line and check for intervals where v is negative, which turns out to be $t < \frac{1}{5}$ or $t > \frac{5}{2}$. If you're wondering where the 0 came from in the answer, that's the given domain.

13. A. This can either be differentiated implicitly, or it can be solved for y. I'll use the product rule. $x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} = 0$, and substituting the given values makes that $x \cdot 8 + 6 \cdot \frac{dx}{dt} = 0$. If we also use the fact that $xy = 18$, since $y = 6$, we also have $x = 3$. That gives $3 \cdot 8 + 6 \cdot \frac{dx}{dt} = 0$, and $\frac{dx}{dt} = -\frac{24}{6} = -4$. That means x is decreasing by 4 units per second.

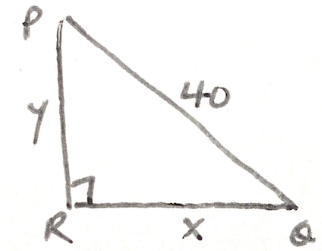
14. D. If you look at that expression for y, you'll see that it seems annoying to differentiate. But we have a calculator, so we can exploit that. $\frac{dy}{dt} = \frac{d}{dx}\left(\frac{15}{x^2 + 1.3^x}\right) \cdot \frac{dx}{dt}$. We know that $\frac{dx}{dt} = 3$, and if $y = 2$, solving the equation $2 = \frac{15}{x^2 + 1.3^x}$ gives that $x \approx 2.37398$. The derivative $\frac{d}{dx}\left(\frac{15}{x^2 + 1.3^x}\right)\Bigg|_{x=2.37398} \approx -1.39655$. Then $\frac{dy}{dt} = -1.39655 \cdot 3 \approx -4.18965$.

15. A. Differentiating both sides with respect to t gives $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{x \cdot \frac{dh}{dt} - h \cdot \frac{dx}{dt}}{x^2}$. That's kind of a mess, but since h is a constant, $\frac{dh}{dt} = 0$. Also, we can rewrite that trig function $\sec^2\theta = \frac{1}{\cos^2\theta}$, and using the Pythagorean theorem on that triangle to the right gives that the hypotenuse is $\sqrt{x^2 + h^2}$. Then $\sec^2\theta = \left(\frac{\sqrt{x^2 + h^2}}{x}\right)^2 = \frac{x^2 + h^2}{x^2}$. Now we've got



$$\frac{x^2 + h^2}{x^2} \cdot \frac{d\theta}{dt} = \frac{0 - h \cdot \frac{dx}{dt}}{x^2}, \text{ and } \frac{d\theta}{dt} = -\frac{h}{x^2} \cdot \frac{x^2}{x^2 + h^2} \cdot \frac{dx}{dt} = -\frac{h}{x^2 + h^2} \cdot \frac{dx}{dt}.$$

16. E. I think it's easier to work with standard variable names, so I've labeled the horizontal side x and the vertical y. For this related rates question, I hope it's clear that the Pythagorean theorem will be useful; you've seen problems like this several times before. We know $x^2 + y^2 = 40^2$, so $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ (as the right side is a constant). The fact that Q is moving $\dfrac{3}{4}$ as fast as P means that $\dfrac{dx}{dt} = -\dfrac{3}{4}\dfrac{dy}{dt}$.



That negative sign comes about because x is increasing as y decreases. Making that substitution gives $2x \cdot -\dfrac{3}{4}\dfrac{dy}{dt} + 2y\dfrac{dy}{dt} = 0$. Even with a calculator to use, that doesn't seem like enough help, as there are still three unknowns in that equation. However, factoring out the 2 and the $\dfrac{dy}{dy}$ will be helpful. That gives $2\dfrac{dy}{dt}\left(-\dfrac{3}{4}x + y\right) = 0$, which will have solutions if either $\dfrac{dy}{dy} = 0$ or if $-\dfrac{3}{4}x + y = 0$. Since the ladder is known to be slipping down the wall, $\dfrac{dy}{dy} \neq 0$. To solve the second equation, $-\dfrac{3}{4}x + y = 0$, we can also exploit the Pythagorean relationship; if $x^2 + y^2 = 40^2$, then $y = \sqrt{40^2 - x^2}$. That means our equation is now $-\dfrac{3}{4}x + \sqrt{40^2 - x^2} = 0$, and the calculator solves this to give $x = 32$. That is an unusually complicated ladder problem, for sure.

17. C. $A = \pi r^2$, so $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. Since $\dfrac{dr}{dt} = 0.2$ and the circumference is already $20\pi = 2\pi r$, this gives $\dfrac{dA}{dt} = 20\pi \cdot 0.2 = 4\pi$.

18. D. From the previous question, we have $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. Since $C = 2\pi r$, $\dfrac{dC}{dt} = 2\pi\dfrac{dr}{dt}$. So if $\dfrac{dA}{dt}$ is twice $\dfrac{dC}{dt}$, the equation is $2\pi r\dfrac{dr}{dt} = 2 \cdot 2\pi\dfrac{dr}{dt}$, and $r = 2$.

19. (a) Since the diameter of the conical container is the same as its total height, similar triangles mean that the diameter is the same as the height all the time. That means when $h = 5$, $d = 5$, and $r = 2.5$. That makes the volume $V = \dfrac{1}{3}\pi \cdot 2.5^2 \cdot 5$ cubic centimeters.

(b) It would be easier to differentiate the volume formula if we simplify it first. As in (a), the diameter is the same as the height, so $r = \dfrac{1}{2}h$, and we can rewrite the volume as $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi \cdot \left(\dfrac{1}{2}h\right)^2 \cdot h = \dfrac{1}{3}\pi \cdot \dfrac{1}{4}h^2 \cdot h = \dfrac{1}{12}\pi h^3$. Differentiating this new volume formula with respect to t gives $\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$. Then $\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{12}\pi \cdot 3 \cdot 5^2 \cdot -\frac{3}{10}$ cm³/h.

(c) We know that $\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$, and since $r = \frac{1}{2}h$, it's also true that $h = 2r$. Substituting that into the expression, $\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3(2r)^2 \cdot \frac{dh}{dt} = \frac{1}{12}\pi \cdot 3 \cdot 4r^2 \cdot \frac{dh}{dt} = \pi r^2 \frac{dh}{dt}$. The rate of change of volume is the area of the circular surface times the constant $\frac{dh}{dt} = -\frac{3}{10}$, and that's the constant of proportionality.

20. (a) This is just the Pythagorean theorem: the distance is $\sqrt{3^2 + 4^2} = 5$.

(b) Let the hypotenuse be z. Then $x^2 + y^2 = z^2$. Differentiating with respect to t gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$. As ship A is traveling *toward* the lighthouse, $\frac{dx}{dt} = -15$, and because ship B is traveling *away*, its derivative is $\frac{dy}{dt} = 10$. Making all of those substitutions, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$ becomes $2(4)(-15) + 2(3)(10) = 2(5)\frac{dz}{dt}$, and that gives $\frac{dz}{dt} = \frac{2(4)(-15) + 2(3)(10)}{2(5)}$, which works out to –6 km/h if you'd care to do that calculation.

(c) While you could use any of the trig functions here, I'm going with tangent. The angle θ is related to the sides x and y by $\tan\theta = \frac{x}{y}$, so $\sec^2\theta\frac{d\theta}{dt} = \frac{y\frac{dx}{dt} - x\frac{dy}{dt}}{y^2}$. Recall that secant is the reciprocal of cosine, so $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{4}$. Then the substitution:

$\left(\frac{5}{4}\right)^2\frac{d\theta}{dt} = \frac{3 \cdot -15 - 4 \cdot 10}{4^2}$, and $\frac{d\theta}{dt} = \frac{\frac{3 \cdot -15 - 4 \cdot 10}{4^2}}{\left(\frac{5}{4}\right)^2}$. No need to work that out.

21. A. Straightforward. The point is (5, 3) and the slope is 4. That makes the tangent line $y - 3 = 4(x - 5)$, so the estimated value of $f(4.8) \approx 4(4.8 - 5) + 3 = 4(-0.2) + 3 = 2.2$.

22. (a) The point we need is given as (12, 36). The slope will be $\left.\frac{dB}{dt}\right|_{t=12} = \frac{5}{\sqrt{12 + 4}} + \frac{1}{9}(12 - 6)^2 = \frac{5}{4} + \frac{1}{9} \cdot 36 = 1.25 + 4 = 5.25$. It's not required to work that out, but I think it's a bit easier to deal with. That makes the required tangent line $y - 36 = 5.25(t - 12)$. We approximate the value as $B(10) \approx 5.25(10 - 12) + 36$ hundred books. Note that the "hundred" there is important. If you work out the arithmetic, you get

$-10.5 + 36 = 25.5$ hundred, or 2550 books. The markscheme actually gives the answer as 2550.

(b) $B''(t) = \frac{d}{dt}\left(\frac{dB}{dt}\right) = \frac{d}{dt}\left(\frac{5}{\sqrt{t+4}} + \frac{1}{9}(t-6)^2\right) = \frac{d}{dt}\left(5(t+4)^{-1/2} + +\frac{1}{9}(t-6)^2\right)$ $= -\frac{5}{2}(t+4)^{-3/2} \cdot 1 + \frac{2}{9}(t-6) \cdot 1$, so $B''(5) = -\frac{5}{2} \cdot 9^{-3/2} + \frac{2}{9} \cdot -1$. That value is clearly negative, which is important to the interpretation. Five hours after the books go on sale, the rate at which books are sold is decreasing at this many hundred books per hour per hour. (Yes, it's lazy to say "this many," but it's truly not necessary to write that expression again, or to work it out. Also, notice the second "per hour" in the units.)

(c) For $B(t)$, we integrate. $B(t) = \int \left(5(t+4)^{-1/2} + \frac{1}{9}(t-6)^2\right) dt$ $= 5 \cdot \frac{(t+4)^{1/2}}{1/2} + \frac{1}{9} \cdot \frac{(t-6)^3}{3} + C$. The initial condition of $B(12) = 36$ allows us to find that constant: $36 = 5 \cdot \frac{(12+4)^{1/2}}{1/2} + \frac{1}{9} \cdot \frac{(12-6)^3}{3} + C = 5 \cdot 2 \cdot \sqrt{16} + \frac{6^3}{9 \cdot 3} + C$ $= 40 + \frac{6 \cdot 6 \cdot 6}{3 \cdot 3 \cdot 3} + C = 40 + 2 \cdot 2 \cdot 2 + C = 48 + C$, so $C = -12$. That makes the model $B(t) = 5 \cdot \frac{(t+4)^{1/2}}{1/2} + \frac{1}{9} \cdot \frac{(t-6)^3}{3} - 12$, and the predicted number of books sold by $t = 21$ equal to $B(21) = 5 \cdot \frac{25^{1/2}}{1/2} + \frac{1}{27} \cdot 15^3 - 12$ hundred books. Again, no need to work that out. The markscheme tells me it's 16,300 books.

23. D. Because $f'(1) = 3$, the tangent line is $y - 4 = 3(x - 1)$. When $x = 1.2$, this gives $f(1.2) \approx y = 4 + 3(1.2 - 1) = 4.6$.

24. B. The tangent line has slope $f'(2) = 5$ and equation $y - 4 = 5(x - 2)$. When $x = 2.1$, $y \approx 4 + 5(2.1 - 2) = 4.5$.

25. C. The fact that the given slope is negative and the point to be estimated is to the left of the given location means that $g(2.6) > g(3)$, but I'll calculate that value anyway. The point and slope provided make the tangent line $y - 2 = -\frac{3}{4}(x - 3)$, so $g(2.6) \approx -\frac{3}{4}(2.6 - 3) + 2$ $= -\frac{3}{4} \cdot -0.4 + 2 = 0.3 + 2 = 2.3$. Knowing that g is concave down on $(2, 4)$ means that the tangent line lies above the graph, and any approximation will be an overestimate.

26. (a) The tangent line has slope $r'(5) = 2.0$ from the table and $r(5) = 30$ from the given radius. The tangent line is $y - 30 = 2.0(t - 5)$, so $r(5.4) \approx 2.0(5.4 - 5) + 30$. Since we know r is concave down on the relevant interval, the tangent line will give an overestimate, so this is greater than the true value.

(b) Related rates! Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$. Then $\left.\frac{dV}{dt}\right|_{t=5} = 4\pi \cdot 30^2 \cdot 2.0$ cubic feet per minute.

(c) The table has the y-values we need for the Riemann sum.

$\int_0^{12} r'(t)dt \approx (2-0)(4.0) + (5-2)(2.0) + (7-5)(1.2) + (11-7)(0.6) + (12-11)(0.5)$.

While you can do the subtractions in your head if you want, they're going to need to see the sum of five products to get you the points here. During the interval from $t = 0$ to $t = 12$ minutes, the radius of the balloon has increased by approximately this many feet. (Again, notice my "this many" reference to avoid either calculating that or writing it again.)

(d) Since r is concave down, $r'(t)$ is decreasing on the interval, and a right Riemann sum will be an underestimate; the approximation is less than the true value of the integral.

27. (a) Using the two times closest to $t = 10$, we get $A'(10) \approx \frac{A(15) - A(5)}{15 - 5} = \frac{25 - 18}{10} = \frac{7}{10}$ gallons per hour.

(b) The theorem that may guarantee any specific value of the derivative is the Mean Value Theorem. Since A is twice differentiable, it is also continuous, and the MVT will apply. That means there is a value of c between $t = 0$ and $t = 30$ (the bounds specified in this part) at which $A'(c) = \frac{A(30) - A(0)}{30 - 0} = \frac{16 - 10}{30} = \frac{6}{30} = \frac{1}{5}$. Therefore there *is* such a time t.

(c) The absolute maximum will happen at a critical point or an endpoint. Critical points are found where the derivative of G is either zero or undefined, so we start with a derivative. $G'(t) = 5 - \frac{2}{3} \cdot \frac{3}{2}(t+9)^{\frac{1}{2}} \cdot 1 = 5 - \sqrt{t+9}$. That is never undefined on the domain [0, 35], and $G'(t) = 5 - \sqrt{t+9} = 0$ when $\sqrt{t+9} = 5$, and $t = 16$. That's the only critical number of G. I initially started to do the Candidates Test for this, where you substitute the endpoints and critical numbers into the function to compare the output values, but the function is a little uglier than I would like without a calculator to back me up, so I will instead make an argument based on the fact that there is just one critical number. The sign of $G'(t)$ changes from positive to negative at $t = 16$, and G is continuous with only one critical number on the given interval. Therefore the local maximum at $t = 16$ is also the absolute maximum. Note that you have to be really careful with this sort of argument; if I didn't have to *type* all of the calculations with those fractions, I would probably just compute those three y-values!

(d) Linear approximations are overestimates if the function they approximate is concave down, and underestimates when the function being approximated is concave up. To decide that, we need the second derivative of G: $G''(t) = \frac{d}{dt}\left(5 - \sqrt{t+9}\right) = -\frac{1}{2}(t+9)^{-\frac{1}{2}}$, and $G'' < 0$ for all values of $t \in [7, 8]$, so the approximation must be an overestimate because G is concave down there.

28. C. As you might expect, a limit appearing in the unit on applications of differentiation is likely to use L'Hôpital's rule. Although this is a multiple-choice question, I'll do the work as though it were free response, justifying everything. First we check the numerator and denominator limits

separately: $\lim_{x\to\pi}(x+\pi\sec x)=\pi+\pi\sec\pi=\pi-\pi=0$ and $\lim_{x\to\pi}\left(x^2-\pi^2\right)=\pi^2-\pi^2=0$. L'Hôpital's rule applies. $\lim_{x\to\pi}\left(\frac{x+\pi\sec x}{x^2-\pi^2}\right)=\lim_{x\to\pi}\frac{1+\pi\sec x\tan x}{2x}=\frac{1+\pi\sec\pi\tan\pi}{2\pi}$ $\frac{1-\pi\cdot 0}{2\pi}=\frac{1}{2\pi}$.

29. C. First, check the limits of both the numerator and the denominator: $\lim_{x\to 0}\left(\sin x\cos x\right)$ $=\sin 0\cos 0=0\cdot 1=0$, and $\lim_{x\to 0}(x)=0$. We take this to the L'Hôpital.
$\lim_{x\to 0}\frac{\sin x\cos x}{x}=\lim_{x\to 0}\frac{\sin x\cdot -\sin x+\cos x\cdot\cos x}{1}=\sin 0\cdot -\sin 0+\cos 0\cdot\cos 0$
$=0+1=1$

30. C. This sort of limit has become more common in recent years, where either the numerator or denominator limit uses the Fundamental Theorem of Calculus to differentiate a function defined as an integral. Again, checking the numerator and denominator limits gives $\lim_{x\to 0}\int_0^x\sin(3t^2)dt$ $=\int_0^0\sin(3t^2)dt=0$ and $\lim_{x\to 0}(4x^3)=0$, so L'Hôpital's rule gives $\lim_{x\to 0}\frac{\int_0^x\sin(3t^2)dt}{4x^3}$ $=\lim_{x\to 0}\frac{\sin(3x^2)}{12x^2}$. That again is of the type $\frac{0}{0}$. I won't bother with the separate limits this time, since it is multiple-choice. $\lim_{x\to 0}\frac{\sin(3x^2)}{12x^2}=\lim_{x\to 0}\frac{\cos(3x^2)\cdot 6x}{24x}=\lim_{x\to 0}\frac{\cos(3x^2)}{4}=\frac{\cos 0}{4}=\frac{1}{4}$

31. B. Exponentials grow faster than everything except factorials and functions with variables as both the base and the exponent (like x^x).

32. D. Either recognize this as the derivative of e^x at $x = 1$, or use L'Hôpital's rule.
$\lim_{h\to 0}\frac{e^{1+h}-e}{h}=\lim_{h\to 0}\frac{e^{1+h}}{1}=\frac{e}{1}=e$