Worked solutions to Unit 3 (Differentiation: Composite, Implicit, and Inverse Functions) problems from notes

1. D. The derivative of f is $f'(x) = \begin{cases} -a \sin x, & x \le 0 \\ b \cos(x + c\pi), & x > 0 \end{cases}$. Both of the parts are differentiable everywhere, so we only have to figure out what happens at $x = 0$. From the left, $\lim_{x \to 0^-} f'(x) = a \sin 0 = 0$ and from the right, $\lim_{x \to 0^+} f'(x) = b \cos(c\pi)$. So it's just necessary to find the set of values that make $b \cos(c\pi)$ *not* equal to 0. A and B are out, since $b = 0$. The cosine of odd multiples of $\pi/2$ is also zero, which means C and E also are out. D gives $1 \cos \pi = -1$, which isn't zero, so that's the answer.

2. B. Use the quotient rule to get $\frac{dy}{dx} = \frac{(x+1)e^{-x} \cdot -1 - e^{-x} \cdot 1}{(x+1)^2} = \frac{e^{-x}(-x-2)}{(x+1)^2}$. Then $\left.\frac{dy}{dx}\right|_{x=1} = \frac{e^{-1}(-3)}{2^2} = -\frac{3}{4e}$.

3. A. While you can use the chain rule here, it's actually easier if you remember how logarithms work. $f(x) = \ln(e^{2x}) = 2x$. Therefore $f'(x) = 2$. Done.

4. E. This one needs the product and chain rules.
$f'(x) = (2x+1) \cdot 4(x^2-3)^3 \cdot 2x + (x^2-3)^4 \cdot 2 = (x^2-3)^3\big((2x+1) \cdot 4 \cdot 2x + (x^2-3) \cdot 2\big)$
$= (x^2-3)^3(16x^2 + 8x + 2x^2 - 6) = 2(x^2-3)^3(9x^2 + 4x - 3)$

5. E. Just a basic chain rule here. $y' = 2(x^3+1)^1 \cdot 3x^2 = 6x^2(x^3+1)$

6. B. This is a chain rule. $B'(x) = g'(f(x))f'(x)$, and then $B'(-3) = g'(f(-3))f'(-3)$. From the graph, $f(-3) = 1$ and $f'(-3) = \frac{1}{3}$ by counting the slope. That makes $B'(-3) = g'(1) \cdot \frac{1}{3}$. Then the graph of $g(x)$ gives that $g'(1) = -\frac{1}{2}$. That's what the tangent lines were about; they let you count the slope of $g(x)$ in a couple of different places. Finally, $B'(-3) = -\frac{1}{2} \cdot \frac{1}{3} = -\frac{1}{6}$.

7. C. First rewriting, $f(x) = \big(\cos(3x)\big)^{2/3}$. That's going to take two chain rules:
$f'(x) = \frac{2}{3}\big(\cos(3x)\big)^{-1/3} \cdot -\sin(3x) \cdot 3 = \frac{-2\sin(3x)}{\sqrt[3]{\cos(3x)}}$.

8. B. Another chain rule here. It can help to rewrite $f(x)$ as $\big(\cos(4x)\big)^3$ to begin with.
$f'(x) = 3 \cdot (\cos(4x))^2 \cdot -\sin(4x) \cdot 4 = -12\cos^2(4x)\sin(4x)$

9. E. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$, so with the chain rule, $\frac{d}{dx}\big(\sin^{-1}(5x)\big) = \frac{1}{\sqrt{1-(5x)^2}} \cdot 5$.

10. B. $P'(x) = f'\left(x^3\right) \cdot 3x^2$, so $P'(1) = f'\left(1^3\right) \cdot 3 \cdot 1^2 = 2 \cdot 3 = 6$

11. A. Use implicit differentiation. $1 - \sin(x+y)\left(1 + \frac{dy}{dx}\right) = 0$, so
$1 + \frac{dy}{dx} = \frac{-1}{-\sin(x+y)} = \csc(x+y)$, and then $\frac{dy}{dx} = \csc(x+y) - 1$.

12. A. Differentiate both sides implicitly with respect to x, then solve for $\frac{dy}{dx}$.

$$x^2\frac{dy}{dx} + y(2x) - 3 = 3y^2\frac{dy}{dx}$$
$$x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 3 - 2xy$$
$$\frac{dy}{dx} = \frac{3-2xy}{x^2-3y^2}$$

So $\left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{3-2(-1)(2)}{(-1)^2 - 3(2)^2} = \frac{7}{-11}$.

13. D. First differentiate everything with respect to x: $2x - \left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + 2y \cdot \frac{dy}{dx} = 0$. Then distribute the negative, collect like terms, and solve for $\frac{dy}{dx}$.

$$2x - x \cdot \frac{dy}{dx} - y + 2y \cdot \frac{dy}{dx} = 0$$
$$2x - y = x \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} = \frac{dy}{dx}(x - 2y)$$
$$\frac{dy}{dx} = \frac{2x-y}{x-2y}$$

The first thing to check is whether this derivative exists at $(2, 1)$. $\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{2 \cdot 1 - 1}{2 - 2 \cdot 1} = \frac{1}{0}$.
Therefore we are down to either D or E. The choices are just horizontal or vertical tangent line, and that slope isn't zero, so the tangent is not horizontal. In general, a slope that's $\frac{\text{nonzero}}{0}$ is going to give a vertical tangent line.

14. B. More implicit differentiation: $\frac{1}{2x+y} \cdot \left(2 + \frac{dy}{dx}\right) = 1$. Multiplying by that denominator on both sides gives $2 + \frac{dy}{dx} = 1(2x+y) = 2x + y$, so $\frac{dy}{dx} = 2x + y - 2$.

15. (a) Once again, we start with implicit differentiation: $2x + 8y \cdot \frac{dy}{dx} = 3\left(x \cdot \frac{dy}{dx} + y \cdot 1\right)$.

Distributing, $2x + 8y \cdot \frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$. Collecting like terms, $8y \cdot \frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2x$. Factoring, $\frac{dy}{dx}(8y - 3x) = 3y - 2x$. And, finally, dividing: $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) If the tangent line is horizontal, then $\frac{dy}{dx} = 0$, and $3y - 2x = 0$, so $y = \frac{2}{3}x$. Substitute that back into the original equation to see if there are any points we can find on the curve where this slope is zero.

$x^2 + 4\left(\frac{2}{3}x\right)^2 = 7 + 3x \cdot \frac{2}{3}x$. Doing some algebra here gives $x^2 + 4 \cdot \frac{4}{9}x^2 = 7 + 2x^2$ and then $\frac{25}{9}x^2 - 2x^2 = 7$, so $\frac{7}{9}x^2 = 7$, and $x^2 = 9$. In fact, that does lead to $x = 3$, and that's the first part of the task. The y-coordinate is easy, since we already have an equation for y: $y = \frac{2}{3}x = \frac{2}{3} \cdot 3 = 2$. (Note that you can do these in the opposite order once you have the numerator of the derivative equal to 0; you can first find the y-coordinate that would make $(3, y)$ produce a slope of 0, then substitute that into the original curve to show that the resulting point is on it.)

(c) More implicit differentiation:

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3y - 2x}{8y - 3x}\right) = \frac{(8y - 3x)\left(3\frac{dy}{dx} - 2\right) - (3y - 2x)\left(8\frac{dy}{dx} - 3\right)}{(8y - 3x)^2}$. There is no need to simplify that before using the point P and the fact that $\frac{dy}{dx} = 0$ there.

$\left.\frac{d^2y}{dx^2}\right|_{(3,2)} = \frac{(8 \cdot 2 - 3 \cdot 3)(3 \cdot 0 - 2) - (3 \cdot 2 - 2 \cdot 3)(8 \cdot 0 - 3)}{(8 \cdot 2 - 3 \cdot 3)^2}$. This is an acceptable answer for the value of the second derivative, but to answer whether this is the location of a maximum or minimum requires determining whether this number is positive or negative. The denominator is the square of a nonzero value, so that is definitely positive. The numerator becomes $(16 - 9)(-2) - 0(-3) = -14$. Therefore $\left.\frac{d^2y}{dx^2}\right|_{(3,2)} < 0$, and the second derivative test tells us that the curve has a local maximum. You can visualize this as a place with a horizontal tangent where the concavity is negative. Again, it was *not* necessary to work out that ugly fraction, but if you did, it comes out to $-\frac{14}{49} = -\frac{2}{7}$.

16. D. Since 4 is an x-value for g and g is the inverse of f, it must be true that 4 is a y-value for f. The slopes of inverse functions are reciprocals, so I need the fact that $f'(x) = 3x^2 + 2x + 1$ and an x-value to substitute in there. If $f(x) = 4 = x^3 + x^2 + x + 1$, x = 1 by inspection. (Seriously, just look at it.) Then $f'(1) = 6$, and $g'(4) = \frac{1}{6}$.

17. A. We're looking for $g'(3)$, so 3 is an x-coordinate of g, and the point (3, __) is on g. Therefore its inverse, $f(x)$, will have a point (__, 3). Based on the list of values we have, when the y-coordinate of f is 3, its x-coordinate is 6 and the point on f is (6, 3). Slopes of inverse functions are reciprocals, so $g'(3) = \dfrac{1}{f'(6)} = \dfrac{1}{-2}$.

18. D. Again, slopes of inverse functions are reciprocals, because inverses switch x- and y-coordinates. We have an x-coordinate of g as 4, so that's the y-coordinate of f. In the table, when the output of f is 4, the input is –2, so $g'(4) = \dfrac{1}{f'(-2)} = \dfrac{1}{4}$.

19. E. All of that bit about $f(g(x))$ and $g(f(x))$ both being equal to x makes f and g inverse functions. The input for g is 8, so that is the output of f. Since $f(3) = 8$, $g(8) = 3$. The slopes are reciprocals, so $g'(8) = \dfrac{1}{f'(3)} = \dfrac{1}{9}$.

20. D. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{d}{dx}\left(\arcsin x\right) = \dfrac{1}{\sqrt{1-x^2}}$, so

$$f'\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}} = \frac{1}{\sqrt{1-\frac{3}{4}}} = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{1}{2}} = 2.$$

21. E. $\dfrac{d}{dx}\left(\tan^{-1} x + 2\sqrt{x}\right) = \dfrac{d}{dx}\left(\arctan x + 2x^{1/2}\right) = \dfrac{1}{1+x^2} + \dfrac{1}{2}x^{-1/2}$. Of the six inverse trig functions, I believe you should only need to have the derivatives of arcsin x and arctan x memorized (and, of course, to recognize the "–1" versions of the notation for those).

22. A. This one uses the chain rule with an inverse trig function. If you think of $\dfrac{x}{2}$ as $\dfrac{1}{2}x$, the derivative of the inside is easy: $y' = \dfrac{d}{dx}\left(\arcsin \dfrac{x}{2}\right) = \dfrac{d}{dx}\left(\arcsin \dfrac{1}{2}x\right) = \dfrac{1}{1+\left(\frac{1}{2}x\right)^2} \cdot \dfrac{1}{2}$.

While it is possible to simplify that, what we really need is a numerical value for the slope at (0, 0), so I'll just go ahead and substitute. When $x = 0$, $y' = \dfrac{1}{1+0^2} \cdot \dfrac{1}{2} = \dfrac{1}{2}$. Only one of those choices has this slope. To get that one, point-slope form gives $y - 0 = \dfrac{1}{2}(x - 0)$, so $y = \dfrac{1}{2}x$ and $2y = x$. Setting that equal to zero gives $x - 2y = 0$. Done.

23. D. $2x + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \dfrac{-2x}{2y} = -\dfrac{x}{y}$. Then $\dfrac{d^2y}{dx^2} = \dfrac{y(-1) - (-x)\frac{dy}{dx}}{y^2} = \dfrac{-y + x\left(-\frac{x}{y}\right)}{y^2}$.

I could pretty it up, but I'm just going to substitute 0 for x anyway. At $(0, 5)$,
$$\frac{d^2y}{dx^2} = \frac{-5 + 0\left(-\frac{0}{5}\right)}{5^2} = -\frac{1}{5}.$$

24. A. Two derivatives here. First $f'(x) = 2\left(\ln x\right)^1 \cdot \frac{1}{x} = \frac{2\ln x}{x}$. Then, using the quotient rule, $f''(x) = \frac{x \cdot 2 \cdot \frac{1}{x} - 2\ln x \cdot 1}{x^2} = \frac{2 - 2\ln x}{x^2}$. Then evaluating that, $f''\left(\sqrt{e}\right) = \frac{2 - 2\ln\left(\sqrt{e}\right)}{\left(\sqrt{e}\right)^2}$.

    Recall that the natural log asks what power goes on e to make the argument, so $\ln\left(\sqrt{e}\right) = \ln\left(e^{1/2}\right) = \frac{1}{2}$. That means $f''\left(\sqrt{e}\right) = \frac{2 - 2 \cdot \frac{1}{2}}{e} = \frac{1}{e}$.

25. D. The first derivative needs a chain rule, and the second derivative adds a product rule to that. Because of how the chain rule multiplies by something, a second derivative like this often needs a product rule. First, $\frac{dy}{dx} = e^{x^3} \cdot \frac{d}{dx}(x^3) = e^{x^3} \cdot 3x^2$. Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{x^3} \cdot 3x^2\right)$ $= e^{x^3} \cdot 6x + 3x^2 \cdot e^{x^3} \cdot 3x^2 = e^{x^3}(6x + 9x^4)$.

26. A. Here, we're looking for a pattern.
    $\frac{dy}{dx} = \frac{d}{dx}(e^{nx}) = e^{nx} \cdot n$
    $\frac{d^2y}{dx^2} = \frac{d}{dx}(e^{nx} \cdot n) = (e^{nx} \cdot n) \cdot n = e^{nx} \cdot n^2$
    $\frac{d^3y}{dx^3} = \frac{d}{dx}(e^{nx} \cdot n^2) = (e^{nx} \cdot n^2) \cdot n = e^{nx} \cdot n^3$
    By now, you should be able to see why A is the right answer.

27. A. Wow, third derivative. Notice the patterns in the answer choices.
    $\frac{d}{dx}\left(\cos x - \ln(2x)\right) = -\sin x - \frac{1}{2x} \cdot 2 = -\sin x - x^{-1}$
    $\frac{d}{dx}\left(-\sin x - x^{-1}\right) = -\cos x + x^{-2}$; interesting that the second derivative was simpler; no chain.
    $\frac{d}{dx}\left(-\cos x + x^{-2}\right) = -(-\sin x) - 2x^{-3} = \sin x - \frac{2}{x^3}$

28. D. Much like #25, this is chain, then product and chain.
    $\frac{dy}{dx} = e^{2\sin x} \cdot 2\cos x$
    $\frac{d^2y}{dx^2} = e^{2\sin x} \cdot -2\sin x + 2\cos x \cdot \left(e^{2\sin x} \cdot 2\cos x\right)$. In product rule problems with e^{stuff}, you can factor that term out of the result, and doing that gives us
    $\frac{d^2y}{dx^2} = e^{2\sin x} \cdot \left(-2\sin x + 4\cos^2 x\right)$. Then factoring out the 2 gives choice D.