Worked solutions to Unit 2 (Differentiation: Definition and Fundamental Properties) problems from notes

1. B. Choice A is true because f is continuous. Choice B is the alternate formula for the definition of the derivative, and it says that the slope of f is 0 at $x = 0$; that's not true because of the corner. Choice C is very similar, but because of the symmetry, the numerator really is 0 for h close to, but not equal to, 0. Choice D is that definition of the derivative formula again, but it says that $f'(2) = -1$, which is true. And Choice E says that $f'(1)$ doesn't exist — because of the corner, that's true.

2. A. This can be looked at as the definition of a derivative. It shows the derivative of cosine at $x = \frac{3\pi}{2}$. That's $-\sin\left(\frac{3\pi}{2}\right) = -(-1) = 1$. Or you could use L'Hôpital's rule.

3. D. Either recognize this as the derivative of e^x at $x = 1$, or use L'Hôpital's rule. For the first one, $\left.\frac{d}{dx}(e^x)\right|_{x=1} = e^1 = e$. For the second, $\lim_{h\to 0}\frac{e^{1+h} - e}{h} = \lim_{h\to 0}\frac{e^{1+h}}{1} = \frac{e}{1} = e$.

4. E. The is the definition of derivative with π where x would ordinarily go. That definition says $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$, so $g'(x) = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}$, and $g'(\pi) = \lim_{h\to 0}\frac{e^{\pi+h} - e^\pi}{h}$.

5. B. This is the definition of the derivative of $\sqrt[3]{x}$ at $x = 8$: $f'(8) = \lim_{h\to 0}\frac{f(8+h) - f(8)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{8+h} - \sqrt[3]{8}}{h} = \lim_{h\to 0}\frac{\sqrt[3]{8+h} - 2}{h}$, so $\left.\frac{d}{dx}\left(\sqrt[3]{x}\right)\right|_{x=2} = \left.\frac{1}{3}x^{-2/3}\right|_{x=2} = \frac{1}{3}\cdot\frac{1}{8^{2/3}} = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$. Or you could use L'Hôpital's rule.

6. B. That's the "alternate" definition of derivative of f at $x = 0$. But there's a corner at $x = 0$, so the derivative doesn't exist, and can't be equal to 0.

7. C. Use the two points closest to $x = 2$, which means the last two. $f'(2) \approx \frac{4.00 - 4.10}{2.00 - 1.98} = \frac{-0.10}{0.02} = -5$

8. B. There really should be a vertical scale, but since there's not, we'll assume that each vertical tick mark is 1. Just count the slope of the tangent line the best you can. It's a little less than 1.

9. D. Least in this case means most negative, and the slope is steepest downward somewhere near $x = 1$.

10. A. Average rate of change is $\dfrac{\Delta y}{\Delta x} = \dfrac{\cos\left(2 \cdot \frac{\pi}{2}\right) - \cos 0}{\frac{\pi}{2} - 0} = \dfrac{\cos \pi - \cos 0}{\frac{\pi}{2}} = \dfrac{-1-1}{\frac{\pi}{2}}$
$= -2 \cdot \dfrac{2}{\pi} = -\dfrac{4}{\pi}$.

11. B. The average rate of change is ordinary slope:
$\dfrac{\Delta y}{\Delta x} = \dfrac{f(10) - f(0)}{10 - 0} = \dfrac{-20-5}{10} = -\dfrac{25}{10} = -\dfrac{5}{2}$.

12. The best approximation of the instantaneous rate of change will use the values as close as you can get to $t = 10$. In this table, that would be $t = 8$ and $t = 12$. Therefore $H'(10) \approx \dfrac{H(12) - H(8)}{12 - 8} = \dfrac{80 - 73}{4} = \dfrac{7}{4}$ °C per minute. The units come from the numerator units divided by the denominator units.

13. (a) This requires the use of a difference quotient:
$S''(10) \approx \dfrac{S(12) - S(8)}{12 - 8} = \dfrac{1.7 - 1.9}{4} = -0.05$. This says that the rate of change of the depth of snow is decreasing at 0.05 cm per hour per hour at $t = 10$.

(b) Since S is twice differentiable, S' is both differentiable and continuous. By the IVT, since $S'(3) = 2.1 > 2 > 1.9 = S'(8)$, there *is* a time of t on $3 < t < 8$ (and therefore on $0 < t < 14$) at which the depth of snow is changing at 2 cm per hour.

(c) The point is $(8, 45)$ and the slope is $S'(8) = 1.9$, so the tangent line is $y = 1.9(t - 8) + 45$. The approximation using this line is $S(10) \approx 1.9(10 - 8) + 45 = 48.8$ cm. Since we know that S is concave down, the tangent line lies above the curve, and this must be an overestimate of the actual depth of the snow.

(d) The value of $D'(10)$ should be found with the calculator.

$$\frac{d}{dt}\left(120-92\cdot e^{\frac{-t}{40}}\right)\Big|_{t=10} \qquad 1.79124$$

At $t = 10$ hours, the model says that the depth of the snow is increasing at approximately 1.791 cm per hour.

14. (a) Using the two times closest to $t = 10$, we get $A'(10) \approx \dfrac{A(15) - A(5)}{15 - 5} = \dfrac{25 - 18}{10} = \dfrac{7}{10}$ gallons per hour. Fun fact: the AP Classroom question bank says that fraction is equal to $\dfrac{7}{5}$. I've reported the error.

(b) The theorem that may guarantee any specific value of the derivative is the Mean Value Theorem. Since A is twice differentiable, it is also continuous, and the MVT will apply. That means there is a value of c between $t = 0$ and $t = 30$ (the bounds specified in this part) at which $A'(c) = \dfrac{A(30) - A(0)}{30 - 0} = \dfrac{16 - 10}{30} = \dfrac{6}{30} = \dfrac{1}{5}$. Therefore there *is* such a time t.

(c) The absolute maximum will happen at a critical point or an endpoint. Critical points are found where the derivative of G is either zero or undefined, so we start with a derivative.

$G'(t) = 5 - \frac{2}{3} \cdot \frac{3}{2}(t+9)^{\frac{1}{2}} \cdot 1 = 5 - \sqrt{t+9}$. That is never undefined on the domain [0, 35], and $G'(t) = 5 - \sqrt{t+9} = 0$ when $\sqrt{t+9} = 5$, and $t = 16$. That's the only critical number of G. I initially started to do the Candidates Test for this, where you substitute the endpoints and critical numbers into the function to compare the output values, but the function is a little uglier than I would like without a calculator to back me up, so I will instead make an argument based on the fact that there is just one critical number. The sign of $G'(t)$ changes from positive to negative at $t = 16$, and G is continuous with only one critical number on the given interval. Therefore the local maximum at $t = 16$ is also the absolute maximum. Note that you have to be really careful with this sort of argument; if I didn't have to *type* all of the calculations with those fractions, I would probably just compute those three y-values!

(d) Linear approximations are overestimates if the function they approximate is concave down, and underestimates when the function being approximated is concave up. To decide that, we need the second derivative of G: $G''(t) = \frac{d}{dt}\left(5 - \sqrt{t+9}\right) = -\frac{1}{2}(t+9)^{-\frac{1}{2}}$, and $G'' < 0$ for all values of $t \in [7, 8]$, so the approximation must be an overestimate because G is concave down there.

15. E. We can see that $f(2) = 4(2-1) = 4$, so choice A is out. For continuity, check the one-sided limits to see if those also turn out to be the same as $f(2)$. $\lim_{x \to 2^-} f(x) = 2^2 = 4$ and $\lim_{x \to 2^+} f(x) = 4(2-1) = 4$. Thus the function is continuous, and we're down to B and E to choose from. For differentiability, we need the derivative. For the left side, $f'(x) = 2x$, and for the right side, $f'(x) = 4$. Since those are the same value when $x = 2$, we can conclude that $f'(x) = \begin{cases} 2x, & x < 2 \\ 4, & x \geq 2 \end{cases}$, and the function is both continuous and differentiable at $x = 2$.

16. D. This looks like an absolute value function, but it's not. We need the derivative: $f'(x) = \frac{1}{2}\left(x^2 + 0.0001\right)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + 0.0001}}$. This is defined at $x = 0$, and if III is true, all of them are. If you graph it on a calculator and zoom in a lot at the origin, you'll see the real behavior.

17. D. You have to know what $f(x) = |x|$ looks like for this. Incredibly, you can use a calculator to graph it this time. I'll let you do that. As you can see, there is a corner at the origin, so II is not true, but that graph is continuous everywhere, and it has a minimum at the origin.

18. B. There is a corner at $x = 0$, so f cannot be differentiable there. However, there are no breaks, so A is true. Critical points are locations where *either* $f' = 0$ or f' is undefined, so that corner means C is true. The curve does bottom out at $x = 0$, so D is true. Finally, you can see that the graph is concave down to the left of $x = 0$ and concave up to the right, so that E is also true.

19. E. This one is legitimately tricky. At first glance, all five of the choices seem like they'd be true. Choices A and B are both true because we know f is differentiable, so it's also continuous. both A and B are consequences of that continuity. Choices C and D are both versions of the definition of the derivative, and they both say that $f'(3) = 5$, which we know is true. That leaves only E. What

this comes down to is that the existence of a derivative does *not* tell us that the derivative is continuous. For reasonably “normal” functions, we’d expect E to be true, but it does not have to be. For an example of a function that has a discontinuous derivative, see https://math.stackexchange.com/questions/292275/discontinuous-derivative. The “basic example” there is the one that usually comes up first in discussions of this behavior.

20. A. Starting from the bottom, the y-intercept is the value of $f(0)$, so that would exist. If $f(x)$ is differentiable, it’s also continuous, and part of the definition of continuity at $x = 0$ says that $f(0)$ exists. Therefore the answer is A; if $f(x)$ has a point discontinuity at $x = 0$, then that limit will exist but $f(0)$ does not have to.

21. A. $f(x) = x^{1/2} + 3x^{-1/2}$, so $f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2}$, and $f'(4) = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} - \frac{3}{2} \cdot \frac{1}{4^{3/2}}$ $= \frac{1}{2} \cdot \frac{1}{2} - \frac{3}{2} \cdot \frac{1}{8} = \frac{1}{4} - \frac{3}{16} = \frac{1}{16}$

22. B. $f'(x) = 3x^2 - 2x + 1$, and $f'(2) = 3 \cdot 2^2 - 2 \cdot 2 + 1 = 12 - 4 + 1 = 9$. One line!

23. B. This uses the constant multiple rule and the difference rule. $h'(x) = 3f'(x) - 2g'(x) - 5 \cdot -\sin x = 3f'(x) - 2g'(x) + 5\sin x$. Therefore $h'(0) = 3f'(0) - 2g'(0) + 5\sin 0 = 3 \cdot 3 - 2 \cdot 7 + 5 \cdot 0 = -5$.

24. B. This needs the sum rule and the constant multiple rule. $A'(x) = f'(x) + 2g'(x)$, so $A'(3) = f'(3) + 2g'(3) = 4 + 2 \cdot -1 = 2$

25. C. Quotient rule: $K'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$, so $K'(0) = \frac{5 \cdot 1 - 2 \cdot -4}{5^2} = \frac{13}{25}$

26. D. Another quotient rule, but a little more complicated this time.
$h'(x) = \frac{2f(x) \cdot (f'(x) - g'(x)) - (f(x) - g(x)) \cdot 2f'(x)}{(2f(x))^2}$. Substituting,
$h'(-1) = \frac{2f(-1) \cdot (f'(-1) - g'(-1)) - (f(-1) - g(-1)) \cdot 2f'(-1)}{(2f(-1))^2}$
$= \frac{2 \cdot -2 \cdot (e - (-3)) - (-2 - 4) \cdot 2 \cdot e}{(2 \cdot -2)^2}$
$= \frac{-4(e + 3) + 6(2e)}{(-4)^2} = \frac{-4e - 12 + 12e}{16} = \frac{8e - 12}{16} = \frac{2e - 3}{4}$. I have to say that I really don’t recall where I got this one, but I don’t think it’s likely to be a multiple choice question these days. There’s too many ways to go wrong that aren’t related to calculus.

27. D. Using the product rule, $g'(x) = xf'(x) + f(x) \cdot 1$, so $g'(2) = 2f'(2) + f(2) = 2 \cdot -5 + 3 = -7$. Only D has the right slope.

28. C. By the sum rule, $h'(x) = f'(x) + g'(x)$. Since both $f(x)$ and $g(x)$ are lines, we get their derivatives by counting slopes: $f'(x) = 1$, and $g'(x) = -\frac{2}{3}$. Thus $h'(x) = 1 + \left(-\frac{2}{3}\right) = \frac{1}{3}$.

29. D. This one needs the quotient rule. $\dfrac{dy}{dx} = \dfrac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \dfrac{1 - \ln x}{x^2}$.

30. B. If $f(x) = \sin x$, then $f'(x) = \cos x$. Therefore $f'\left(\dfrac{\pi}{3}\right) = \cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$. Sometimes you need to know values of trig functions.

31. D. This one is a product rule. $h'(x) = (2f(x) + 3) \cdot g'(x) + (1 + g(x)) \cdot 2f'(x)$. That means $h'(1) = (2f(1) + 3) \cdot g'(1) + (1 + g(1)) \cdot 2f'(1) = (2 \cdot 3 + 3) \cdot 4 + (1 + -3) \cdot 2 \cdot -2$
$= 9 \cdot 4 + -2 \cdot -4 = 36 + 8 = 44$

32. A. This is another product rule. First, we calculate $p'(x) = f(x)g'(x) + g(x)f'(x)$. That means $p'(-2) = f(-2)g'(-2) + g(-2)f'(-2)$. In that graph of $f(x)$, we see that $f(-2) = 0$ and $f'(-2) > 0$. From the graph of $g(x)$, we get that $g(-2) < 0$ and $g'(-2) > 0$. Therefore $p'(-2) = 0 \cdot \text{positive} + \text{negative} \cdot \text{positive}$. That's negative, so $p'(-2) < 0$.

33. C. This one is product and a little bit of chain rule. $f'(x) = \ln x \cdot -\sin(4x) \cdot 4 + \cos(4x) \cdot \dfrac{1}{x}$
$= -4 \ln x \sin(4x) + \dfrac{\cos(4x)}{x}$

34. D. The instantaneous rate of change means the derivative, and that's the quotient rule here.
$f'(x) = \dfrac{(x-1) \cdot 2x - (x^2 - 2) \cdot 1}{(x-1)^2}$, so $f'(2) = \dfrac{(2-1) \cdot 2 \cdot 2 - (2^2 - 2)}{(2-1)^2} = \dfrac{4-2}{1} = 2$

35. C. More quotient rule. $f'(x) = \dfrac{(x+3)(2x+3) - (x^2 + 3x + 2)(1)}{(x+3)^2}$
$= \dfrac{2x^2 + 9x + 9 - x^2 - 3x - 2}{(x+3)^2} = \dfrac{x^2 + 6x + 7}{(x+3)^2}$

36. C. All of these start out the same way, by rewriting $\cot x$ as the reciprocal of $\tan x$. To differentiate that needs the quotient rule. If you look at the choices, you'll see that only C has the minus in the middle of the numerator; that's the right answer. However, that does *not* look like the version of the derivative of cotangent that you memorized in the fall. Here's how those are the same, using trig identities: $-\dfrac{\sec^2 x}{\tan^2 x} = -\dfrac{1}{\cos^2 x} \div \dfrac{\sin^2 x}{\cos^2 x} = -\dfrac{1}{\cos^2 x} \cdot \dfrac{\cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$.