Worked solutions to Unit 1 (Limits and Continuity) problems from notes

1. C. I is true because the left-hand limit exists and the dot is filled in. II is true because the two-sided limit exists at $x = 6$, even though that is different from the function value. III is not true because that two-sided limit at $x = 6$ doesn't match the value of the function.

2. A. Since sine is a continuous function, $\lim\limits_{x \to 1} \sin(f(x)) = \sin\left(\lim\limits_{x \to 1} f(x)\right) = \sin(2) \approx 0.909$.

3. C. The left- and right-hand limits both exist, but because those are different, the two-sided limit doesn't exist. Note that only the right-hand limit matches the y-value of the function, but the left-hand limit *does* still exist.

4. C. Not (A) or (B) because 5.95 and 6.999 are about 1 unit apart (6 vs. 7), so the two-sided limit cannot exist. The left-hand limit is the one with the negative sign in the notation, and that's 6 rather than 7.

5. E. Always try plugging in first. In this case, you get $\frac{2}{0}$, which doesn't exist.

6. C. Factor the denominator and reduce the fraction. $\lim\limits_{x \to 5} \frac{x-5}{(x-5)(x+5)} = \lim\limits_{x \to 5} \frac{1}{x+5} = \frac{1}{10}$.

7. D. The little minus sign on the top right of the 2 means this is the limit from the left, so we'll use the first line of the function; $2^2 + 1 = 5$.

8. D. Factor. $\lim\limits_{x \to 0} \frac{x^3(2x^3+6)}{x^3(4x^2+3)} = \lim\limits_{x \to 0} \frac{2x^3+6}{4x^2+3} = \frac{6}{3} = 2$.

9. C. Factor again. $\lim\limits_{x \to 2} \frac{(x-2)(x-k)(x+k)}{(x+2)(x-2)(x-k)} = \lim\limits_{x \to 2} \frac{(x+k)}{(x+2)}$. Then substituting, we have $\lim\limits_{x \to 2} \frac{(x+k)}{(x+2)} = \frac{2+k}{4} = 3$. So $2 + k = 12$ and $k = 10$.

10. A. This is a great place for L'Hôpital's rule, since the limit is of the indeterminate form 0/0 and it's easy to differentiate the numerator and denominator. $\lim\limits_{x \to 0} \frac{\sin 3x}{7x} - \lim\limits_{x \to 0} \frac{3\cos 3x}{7} = \frac{3\cos 0}{7} = \frac{3}{7}$. That technique is not actually part of the course until unit 4, but it seemed like a good place to remind you of it anyway. It can be done with manipulation of the memorized limit $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$, but I don't think you'd want to do it that way.

11. C. The inequality telling that f is between g and h suggests that this may be a squeeze theorem question. Consider the limits of g and h as x approaches 1:

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}\left(\sin\left(\frac{\pi}{2}x\right)+4\right) = \sin\left(\frac{\pi}{2}\right)+4 = 5$$
$$\lim_{x\to 1} h(x) = \lim_{x\to 1}\left(-\frac{1}{4}x^3+\frac{3}{4}x+\frac{9}{2}\right) = -\frac{1}{4}+\frac{3}{4}+\frac{9}{2} = 5$$
Since these are the same, f must share their limit as x approaches 1.

12. B. In addition to an inequality like the ones in I, II, and III, the Squeeze Theorem requires that the limits of the bounding functions be equal to each other. So we should check those limits.
I. $\lim_{x\to 0}\frac{1}{3}\left(1-x^2\right) = \frac{1}{3}$, which is not equal to $\frac{1}{2}$. Therefore I cannot be part of the right answer.
II. $\lim_{x\to 0}\left(-x^2\right) = \lim_{x\to 0}\left(x^2\right) = 0$, so this one is going to work. (Yes, you could have seen that from the fact that (A) and (C) were already eliminated, and both the other answers include II.)
III. Neither $\lim_{x\to 0}\left(-\frac{1}{|x|}\right)$ nor $\lim_{x\to 0}\left(\frac{1}{|x|}\right)$ exist, since they both put 0 in the denominator. Those functions have a shared vertical asymptote at $x = 0$, so nothing is squeezed. Only II works.

13. D. Considering case I, $\lim_{x\to 3^-}(2x+1) = 7$ and $\lim_{x\to 3^+}(10-x) = 7$, so that two-sided limit exists, eliminating choice A. For case II, it appears that $\lim_{x\to 3^-} h(x) = 7$, but $\lim_{x\to 3^+} h(x) = 4$. That two-sided limit does not exist, so the right answer much include II. And III appears to be a graph of the function with the table values in II; the one-sided limits match II, and that two-sided limits also fails to exist.

14. E. We can see that $f(2) = 4(2-1) = 4$, so choice A is out. For continuity, check the one-sided limits to see if those also turn out to be the same as $f(2)$. $\lim_{x\to 2^-} f(x) = 2^2 = 4$ and $\lim_{x\to 2^+} f(x) = 4(2-1) = 4$. Thus the function is continuous, and we're down to B and E to choose from. For differentiability, we need the derivative. For the left side, $f'(x) = 2x$, and for the right side, $f'(x) = 4$. Since those are the same value when $x = 2$, we can conclude that $f'(x) = \begin{cases} 2x, & x < 2 \\ 4, & x \geq 2 \end{cases}$, and the function is both continuous and differentiable at $x = 2$.

15. D. Clearly $f(a)$ exists because it is defined to be 0, so II is true, eliminating A and B. If we factor and reduce the first line, the function becomes $f(x) = \begin{cases} x+a, & x \neq a \\ 0, & x = a \end{cases}$, and $\lim_{x\to a} f(x) = a + a = 2a$, so I is true, too, eliminating C. However, we are given intially that $a \neq 0$, so $2a \neq f(a)$, and III is not true.

16. D. As x approaches 0 from the right, the expression looks like $\frac{1}{(\text{tiny positive})(1)}$, which goes to infinity. That will produce a vertical asymptote.

17. (a) $\frac{g(5)-g(4)}{5-4}=\frac{(-1)-(-3)}{5-4}$. You can work this out to be 2, but that's not necessary.

(b) As approaches 4 from the left and from the right, $f(x)$ approaches 2 from below. Therefore, the limit of g is evaluated as x approaches 2 from the left (the "below 2" side). We have $\lim_{x\to 4} h(x) = \lim_{x\to 2^-} g(x) = 1$.

(c) For a jump discontinuity, we need the left and right limits both to exist, but to have different values so that the two-sided limit does not exist. The $x = 4$ choice can be eliminated, since both $\lim_{x\to 4} f(x)$ and $\lim_{x\to 4} g(x)$ exist. For $x = -1$, $\lim_{x\to -1^-} \left(f(x)+g(x)\right) = 0 + -1 = -1$ and $\lim_{x\to -1^+} \left(f(x)+g(x)\right) = 1 + 1 = 2$. Those are different, so $x = -1$ is the right answer. But why not also $x = 2$? $\lim_{x\to 2^-} \left(f(x)+g(x)\right) = 0 + 1 = 1$ and $\lim_{x\to 2^+} \left(f(x)+g(x)\right) = -2 + 3 = 1$, so the two-sided limit exist at that location. Based on the markscheme from the College Board, you'd get full credit with the limit justification for $x = -1$, ignoring the other two possibilities, likely because the question tells you just one of the values works.

(d) Note that the table refers to the same function as the graph of f at the top of the question. This is a removable (also called "point") discontinuity. The limit would appear to be between 2.615 and 2.621. Halfway is the best guess: 2.618.

18. Yes. The justification requires limits (an answer that doesn't involve limits doesn't earn any marks). $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{x+1} = \sqrt{4} = 2$, and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (5-x) = 5-3 = 2$. Notice that it does not matter at all that x never gets to 3 in the right-hand part of the definition; as x gets incredibly *close* to 3 in the expression $5 - x$, the value of $5 - x$ gets incredibly close to 2. Since those left- and right-hand limits are equal, the two-sided limit exists. However, the definition of continuity also requires that $\lim_{x\to c} f(x) = f(c)$, so to finish the justification, it's $f(3) = 2 = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$. In fact, that last long equation is sufficient justification.

19. A. As above, that's basically the definition of continuity. Choice B cannot be true, since we don't know anything at all about the differentiability of the function; a function can be continuous without being differentiable. Choice C isn't true because we don't know anything about the function anywhere besides where x approaches a. For D, we don't know that $f(x)$ has a tangent line at all, so there's no reason to think it's this particular horizontal line. And finally, we know that $f(x)$ is continuous, and a vertical asymptote is an infinite discontinuity, so E is a no.

20. B. The graph is discontinuous at $x = 0$, 3, and 5, but not at $x = 6$, so we can immediately eliminate D as an option. The reason given there describes the corner at that location and therefore why f isn't *differentiable*. For the other choices, it's necessary to examine the reasons to see if those match the graph. For A, the graph shows that $\lim_{x\to 0} f(x) = 1$, but that $f(0)$ does not exist. That's not the same as saying that the *limit* does not exist, so A is out. For C, the *two*-sided limit at $x = 5$ fails to exist, which causes the discontinuity, but the limit from the *left* is 4, which does exist. That leaves B; $\lim_{x\to 3} f(x) = 0$, but $f(3) = 2$.

21. B. It's undefined when $x = -1$. All of the rest of them are continuous everywhere.

22. C. To be continuous, we need left- and right-hand limits to be equal at $x = 2$. The left side is $\lim_{x\to 2^-}(x^2 - 3x + 9) = 2^2 - 3 \cdot 2 + 9 = 7$, and the right side is $\lim_{x\to 2^+}(kx + 1) = 2k + 1$. So $2k + 1 = 7$, and $k = 3$.

23. A. That function is only undefined when its denominator is zero. Otherwise, it's the quotient of two continuous functions.

24. B. The first interval includes the sections with e^x and $x^3 + 2x$, so we check the location where the definition of f changes from one to the other: $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} e^x = e^{-1}$, but $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(x^3 + 2x) = -3$, so f is not continuous on $(-5, 0)$, and A is out. The second interval includes $x^3 + 2x$ and $\frac{5x}{x-2}$, switching at $x = 0$. Again, checking the two one-sided limits gives $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^3 + 2x) = 0 = f(0)$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{5x}{x-2} = 0$. The only place either of those parts is undefined would be at $x = 2$, which is not included in the interval in B. For C, $\frac{5x}{x-2}$ is not continuous at $x = 2$; for D, the two one-sided limits are different at $x = 4$.

25. B. Each of the two pieces is continuous on its domain, so it only remains to make the two pieces meet up at $x = 1$. That happens when the left and right limits are equal: $\lim_{x\to 1^-}(2 - \sin x) = \lim_{x\to 1^+}(cx\sqrt{x^2+2} + c)$ gives $1 - \sin 1 = c \cdot 1\sqrt{1^2 + 2} + c$, and solving that with a calculator gives $c \approx 0.424051$.

26. B. This has the form ∞/∞, and you could use L'Hôpital's rule, but I think it's easier to recognize that the denominator is the square of the numerator in the first place.

$$\lim_{x\to\infty}\frac{\sqrt{x-2}}{x-2} = \lim_{x\to\infty}\frac{1}{\sqrt{x-2}} \to \frac{1}{\infty} \to 0$$

27. C. Consider the highest powers; as x increases without bound, the other terms pale by comparison.

$$\lim_{x\to\infty}\frac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} = \lim_{x\to\infty}\frac{x^3}{4x^3} = \frac{1}{4}$$

28. B. Consider the highest powers again. $\lim_{x\to\infty}\frac{x^2 - 4}{2 + x - 4x^2} = \lim_{x\to\infty}\frac{x^2}{-4x^2} = \frac{1}{-4}$

29. B. Set the denominator equal to zero to get $x = 1$. If the denominator evaluates to 0 and the numerator does not, that gives a vertical asymptote.

30. D. Horizontal asymptotes are found with the limit as x approaches ∞ or $-\infty$. $\lim_{x\to\infty}\frac{x}{x-1} = 1$, so the horizontal asymptote is the line $y = 1$.

31. D. Tricky. Again, consider the limits as $x \to \infty$ and as $x \to -\infty$. As x increases *or* decreases without bound, the denominator becomes basically $\sqrt{x^2}$, but that's actually the absolute value of x rather than x itself. When x is positive, the numerator and denominator have the same sign, and the limit is 3. When x is negative, they have opposite signs, and the limit is –3.

32. D. If the denominator is going to have $x = 2$ as a zero, it must be $x^2 - 4$, and $b = -4$. Only one of the choices matches that.

33. E. This one is easiest when one considers relative rates of growth. Of the ordinary types of functions we deal with most of the time, exponential functions grow fastest and logarithms grow slowest, with polynomials and other functions that use the power rule in between. For the limit as $x \to \infty$ to be 0, the faster-growing function must be in the denominator. In I, the power function in the denominator beats out the logarithm in the numerator, and in III, the exponential in the denominator grows faster than the power in the numerator.

34. B. Exponentials grow faster than everything except factorials and functions with variables as both the base and the exponent (like x^x).

35. (a) That denominator factors to $x^3 - 4x = x(x+2)(x-2)$, and $f(x)$ is undefined when its denominator is 0, at $x = 0$, 2, and –2, so the domain is all real x except 0, 2, and –2.

(b) It looks like it might be L'Hôpital's rule, so we check the limits of the numerator and denominator separately: $\lim\limits_{x\to 0}\left((x-2)\sin x\right) = -2 \cdot 0 = 0$ and $\lim\limits_{x\to 0}(x^3 - 4x) = 0$. This is of the form 0/0, so we can use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{(x-2)\sin x}{x^3-4x} = \lim_{x\to 0}\frac{(x-2)\cos x + \sin x \cdot 1}{3x^2-4} = \frac{-2+0}{0-4} = \frac{1}{2}$$

(c) Factoring the denominator, we see that the factors of $x - 2$ cancel: $\frac{(x-2)\sin x}{x(x^2-4)} = \frac{(x-2)\sin x}{x(x+2)(x-2)} = \frac{\sin x}{x(x+2)}$. We just determined that the limit exists as x approaches 0. So the only vertical asymptote is $x = -2$, since that makes the denominator 0, but the numerator nonzero.

(d) Consider the simplified version: $\lim\limits_{x\to -2^-}\frac{\sin x}{x(x+2)}$. Because $-\pi < -2 < 0$, the numerator is negative. For x close to –2 from the left, the denominator is positive and close to 0. The limit is therefore $-\infty$. You'd probably get away with saying that it doesn't exist.

36. a) Factor and look for something to cancel: $\lim\limits_{x\to 3}\frac{|x|(x-3)}{(3-x)(3+x)}$. While $x - 3$ and $3 - x$ are not identical, they are opposites, and they will cancel to leave a –1. Then you can substitute to get a value. $\lim\limits_{x\to 3}\frac{|x|(x-3)}{(3-x)(3+x)} = \lim\limits_{x\to 3}\frac{|x|}{-1(3+x)} = \frac{3}{-1\cdot 6} = -\frac{1}{2}$

b) Since the $3 - x$ factor canceled, the only vertical asymptote is $x = -3$, as that will make the denominator 0 while the numerator is nonzero.

c) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{|x|}{-1(3+x)} = \lim\limits_{x\to\infty} \dfrac{x}{-x} = -1$, so $y = -1$ is one horizontal asymptote.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{|x|}{-1(3+x)} = \lim\limits_{x\to-\infty} \dfrac{-x}{-x} = 1$, so $y = 1$ is the other one. That numerator is $-x$ because x is negative, so its absolute value will be the opposite of the negative.

d) The only nonremovable discontinuity is at the vertical asymptote, at $x = -3$.

37. B. The given fact that f is continuous on the given interval makes the Intermediate Value Theorem apply, so $f(x)$ must take on every y-value between –5 and 4, the y-values at the endpoints. That makes choices A and C both true. Continuity also makes the Extreme Value Theorem apply, so f must have both a maximum and a minimum y-value on the closed interval given. Choice E says that there is a biggest y-value, so that's true. The differentiability of f on the open interval means that the Mean Value Theorem also applies. That theorem guarantees a value of c on $-2 < c < 1$ at which $f'(c) = \dfrac{f(1) - f(-2)}{1-(-2)} = \dfrac{4-(-5)}{3} = -3$. That makes D true, and only B can be false.

38. B. If g is continuous on [–1, 2], then it will take on every y-value between $g(-1) = 0$ and $g(2) = 5$, including 3. If g is not continuous, it can still be defined but skip right over that y-value, which is why A and C aren't enough. Knowing that the slope is 5 somewhere also doesn't tell us that g must be continuous everywhere, and neither does the fact that the value of the integral is 3.

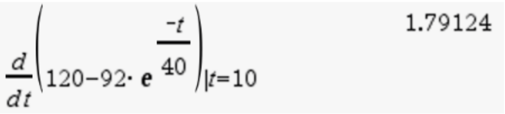
39. D. The Intermediate Value Theorem just requires that f be continuous on the closed interval in question, and we have that. It guarantees that f will hit every y-value between the y-values at the endpoints. That makes A, D, and E all look possible. It's not A, because the theorem doesn't say anything about which x-value this has to happen at. The difference between D and E is the set of boundaries for c. In this problem, –3 and 6 are the x-values, so the answer is D. The values of –1 and 3 in E are values of y, and c is a value of x.

40. (a) This requires the use of a difference quotient:
$S''(10) \approx \dfrac{S(12) - S(8)}{12 - 8} = \dfrac{1.7 - 1.9}{4} = -0.05$. This says that the rate of change of the depth of snow is decreasing at 0.05 cm per hour per hour at $t = 10$.

(b) Since S is twice differentiable, S' is both differentiable and continuous. By the IVT, since $S'(3) = 2.1 > 2 > 1.9 = S'(8)$, there *is* a time of t on $3 < t < 8$ (and therefore on $0 < t < 14$) at which the depth of snow is changing at 2 cm per hour.

(c) The point is $(8, 45)$ and the slope is $S'(8) = 1.9$, so the tangent line is $y = 1.9(t - 8) + 45$. The approximation using this line is $S(10) \approx 1.9(10 - 8) + 45 = 48.8$ cm. Since we know that S is concave down, the tangent line lies above the curve, and this must be an overestimate of the actual depth of the snow.

(d) The value of $D'(10)$ should be found with the calculator.



$\dfrac{d}{dt}\left(120-92\cdot e^{\frac{-t}{40}}\right)\Big|_{t=10}$ 1.79124

At $t = 10$ hours, the model says that the depth of the snow is increasing at approximately 1.791 cm per hour.