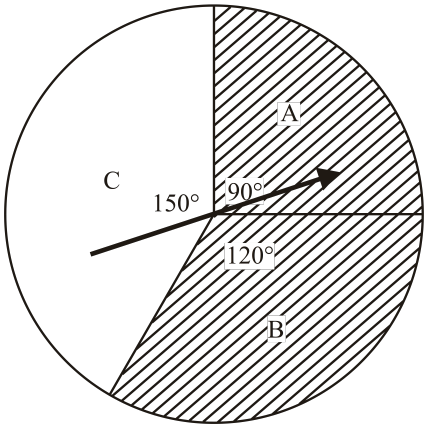
Probability
Worked solutions for practice problems

1. The following diagram shows a circle divided into three sectors A, B and C. The angles at the centre of the circle are 90°, 120° and 150°. Sectors A and B are shaded as shown.



The arrow is spun. It cannot land on the lines between the sectors. Let *A, B, C* and *S* be the events defined by

***A*: Arrow lands in sector A**
***B*: Arrow lands in sector B**
***C*: Arrow lands in sector C**
***S*: Arrow lands in a shaded region.**

Find

(a) P(*B*);

(b) P(*S*);

(c) P($A \mid S$).

(a) There are 360° in the circle. $P(B) = \dfrac{120}{360} = \dfrac{1}{3}$

(b) $\dfrac{90 + 120}{360} = \dfrac{210}{360} = \dfrac{21}{36} = \dfrac{7}{12}$

(c) There are 210° in S and 90° of those are in A. $P(A|S) = \dfrac{90}{210} = \dfrac{9}{21} = \dfrac{3}{7}$

Note that while there is a formula available for conditional probability, that's not the best way to do this problem.

2. For the events *A* and *B*, $p(A) = 0.6$, $p(B) = 0.8$ and $p(A \cup B) = 1$.

Find

(a) $p(A \cap B)$;

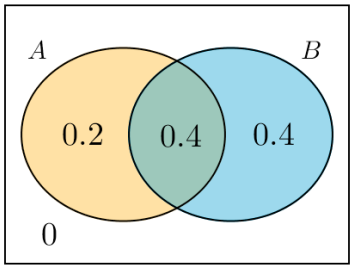
(b) $p(A' \cup B')$.

(a) $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$1 = 0.6 + 0.8 - p(A \cap B)$

$p(A \cap B) = 0.6 + 0.8 - 1 = 0.4$

(b) A Venn diagram can help here.



The complement of A contains just the 0.4 in blue; the complement of B contains only the 0.2 in orange. The union of the complements is therefore 0.2 + 0.4 = 0.6

3. Consider events A, B such that $P(A) \neq 0$, $P(A) \neq 1$, $P(B) \neq 0$, and $P(B) \neq 1$.
In each of the situations (a), (b), (c) below state whether A and B are
mutually exclusive (M);
independent (I);
neither (N).

(a) $P(A|B) = P(A)$
(b) $P(A \cap B) = 0$
(c) $P(A \cap B) = P(A)$

(a) This is the definition of independence: I.
(b) This is the definition of mutual exclusivity: M.
(c) If the events were mutually exclusive, $P(A \cap B)$ would be 0, but we know that $P(A) \neq 0$, so the events cannot be mutually exclusive. If the events were independent, then $P(A \cap B)$ would be equal to $P(A)P(B)$. If $P(A)P(B) = P(A)$, then $P(B) = 1$, and we know that's not true, either. So the events cannot be independent. Therefore they are neither of these: N.

4. Given that events A and B are independent with $P(A \cap B) = 0.3$ and $P(A \cap B') = 0.3$, find $P(A \cup B)$.

The two sets $A \cap B$ and $A \cap B'$ are the two parts of A. That means $P(A) = 0.3 + 0.3 = 0.6$. Because the two events are independent, $P(A \cap B) = P(A)P(B)$.

$0.3 = 0.6\ P(B)$

$$P(B) = \frac{0.3}{0.6} = 0.5$$

Then $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.5 - 0.3 = 0.8$.

5. For events A and B, the probabilities are $P(A) = \frac{3}{11}$, $P(B) = \frac{4}{11}$.

Calculate the value of $P(A \cap B)$ if

(a) $P(A \cup B) = \frac{6}{11}$;
(b) **events A and B are independent.**

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{6}{11} = \frac{3}{11} + \frac{4}{11} - P(A \cap B)$$

$$P(A \cap B) = \frac{1}{11}$$

(b) For independent events, $P(A \cap B) = P(A)P(B) = \frac{3}{11} \cdot \frac{4}{11} = \frac{12}{121}$.

6. The independent events *A, B* are such that P(*A*) = 0.4 and P(*A* ∪ *B*) = 0.88. Find

(a) P(*B*)

(b) the probability that either *A* occurs or *B* occurs, but *not* both.

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

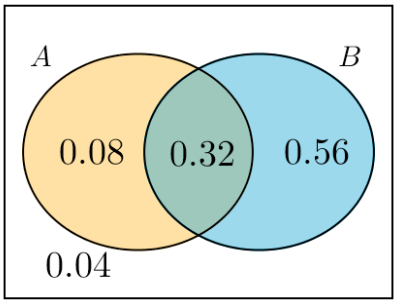
Because the events are independent, $P(A \cap B) = P(A)P(B)$.

$0.88 = 0.4 + P(B) - 0.4\ P(B)$

$0.48 = 0.6\ P(B)$

$P(B) = 0.8$

(b) A Venn diagram seems like the best way to understand this.



$P(A \cap B) = P(A)P(B) = 0.8 \cdot 0.4 = 0.32$ in the center, so the orange part on the left is $0.4 - 0.32 = 0.08$. Similarly, the blue part on the right is $0.88 - 0.32 = 0.56$. Subtracting those three sections from 1 gives 0.04 on the outside. Now either *A* or *B* but not both must be $0.08 + 0.56 = 0.64$.

7. A bag contains 2 red balls, 3 blue balls and 4 green balls. A ball is chosen at random from the bag and is not replaced. A second ball is chosen. Find the probability of choosing one green ball and one blue ball in any order.

Either we get green, then blue, or blue, then green.

$P(\text{green, then blue}) = \frac{4}{9} \cdot \frac{3}{8} = \frac{12}{72} = \frac{1}{6}$

$P(\text{blue, then green}) = \frac{3}{9} \cdot \frac{4}{8} = \frac{12}{72} = \frac{1}{6}$

Since these are mutually exclusive occurrences, the probability of either is $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$.

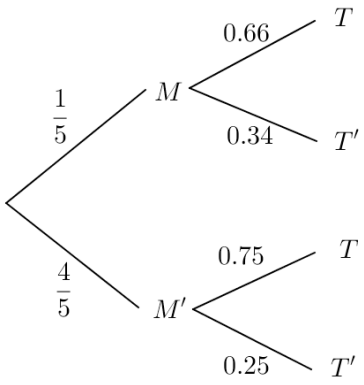
This would also have done nicely with a tree.

8. Robert travels to work by train every weekday from Monday to Friday. The probability that he catches the 08.00 train on Monday is 0.66. The probability that he catches the 08.00 train on any other weekday is 0.75. A weekday is chosen at random.

(a) Find the probability that he catches the train on that day.

(b) Given that he catches the 08.00 train on that day, find the probability that the chosen day is Monday.

(a) A tree diagram seems the best way to organize this. Let *M* represent Monday, *M′* another weekday, *T* that he catches the 8:00 train, and *T′* that he misses it.

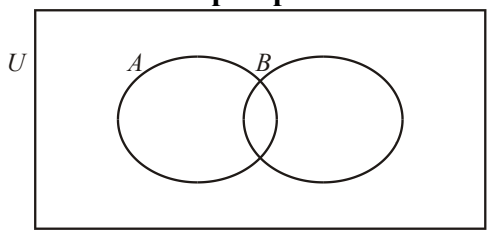


The probability that he catches the train on any day chosen at random is

$$\mathrm{P}(M \cap T) + \mathrm{P}(M \cap T') = \frac{1}{5} \cdot 0.66 + \frac{4}{5} \cdot 0.75 = 0.732.$$

(b) $$\mathrm{P}(T \mid M) = \frac{\mathrm{P}(T \cap M)}{\mathrm{P}(T)} = \frac{\frac{1}{5} \cdot 0.66}{0.732} \approx 0.180.$$

9. The following Venn diagram shows a sample space U and events A and B.



$n(U) = 36$, $n(A) = 11$, $n(B) = 6$ and $n(A \cup B)' = 21$.

(a) On the diagram, shade the region $(A \cup B)'$.

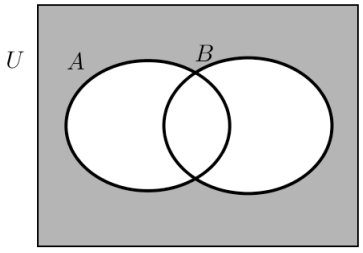
(b) Find

(i) $n(A \cap B)$;

(ii) $\mathrm{P}(A \cap B)$.

(c) Explain why events A and B are not mutually exclusive.

(a) The complement of the union means everything that's outside the union.



(b) (i) $n(A \cup B) = 36 - 21 = 15$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

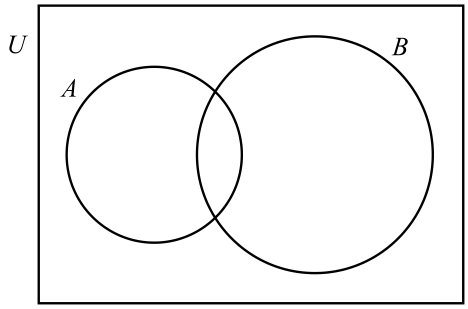
$15 = 11 + 6 - n(A \cap B)$

$n(A \cap B) = 2$

(ii) $$\mathrm{P}(A \cap B) = \frac{n(A \cap B)}{n(U)} = \frac{2}{36} = \frac{1}{18}$$

(c) The events are not mutually exclusive because $\mathrm{P}(A \cap B) \neq 0$.

10. The following Venn diagram shows the universal set U and the sets A and B.



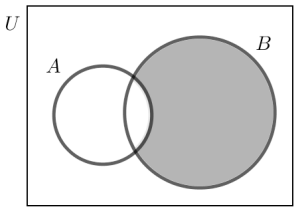
(a) Shade the area in the diagram which represents the set $B \cap A'$.

$$n(U) = 100,\ n(A) = 30,\ n(B) = 50,\ n(A \cup B) = 65.$$

(b) Find $n(B \cap A')$.

(c) An element is selected at random from U. What is the probability that this element is in $B \cap A'$?

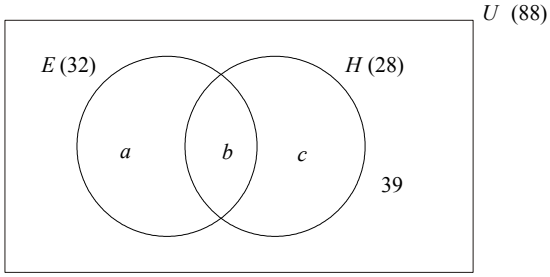
(a) The shading is in the part that's B and at the same time outside of A.



(b) $n(B \cap A') = n(A \cup B) - n(A) = 65 - 30 = 35$

(c) $\mathrm{P}(B \cap A') = \dfrac{n(B \cap A')}{n(U)} = \dfrac{35}{100}$

11. In a school of 88 boys, 32 study economics (E), 28 study history (H) and 39 do not study either subject. This information is represented in the following Venn diagram.



(a) Calculate the values a, b, c.

(b) A student is selected at random.

(i) Calculate the probability that he studies both economics and history.

(ii) Given that he studies economics, calculate the probability that he does not study history.

(c) A group of three students is selected at random from the school.

(i) Calculate the probability that none of these students studies economics.

(ii) Calculate the probability that at least one of these students studies economics.

(a) $88 - 39 = 32 + 28 - b$, so $b = 11$

$a + b = 32$, so $a = 21$

$b + c = 28$, so $c = 17$

(b) (i) $\mathrm{P}(E \cap H) = \dfrac{b}{n(U)} = \dfrac{11}{88} = \dfrac{1}{8}$

(ii) $\mathrm{P}(H' \mid E) = \dfrac{\mathrm{P}(H' \cap E)}{\mathrm{P}(E)} = \dfrac{n(H' \cap E)}{n(E)} = \dfrac{a}{32} = \dfrac{21}{32}$

(c) (i) 88 – 32 = 56 do not study economics.

$\dfrac{56}{88} \cdot \dfrac{55}{87} \cdot \dfrac{54}{86} = \dfrac{315}{1247} \approx 0.253$

(ii) $1 - \dfrac{315}{1247} = \dfrac{932}{1247} \approx 0.747$

12. In a survey of 200 people, 90 of whom were female, it was found that 60 people were unemployed, including 20 males.

(a) Using this information, complete the table below.

	Males	Females	Totals
Unemployed			
Employed			
Totals			200

(b) If a person is selected at random from this group of 200, find the probability that this person is

(i) an unemployed female;

(ii) a male, given that the person is employed.

(a)

	Males	Females	Totals
Unemployed	**20**	**40**	**60**
Employed	**90**	**50**	**140**
Totals	**110**	**90**	200

(b) (i) $\dfrac{n(\text{unemployed females})}{n(\text{people})} = \dfrac{40}{200} = \dfrac{1}{5}$

(ii) $\dfrac{n(\text{employed men})}{n(\text{employed people})} = \dfrac{90}{140} = \dfrac{9}{14}$

13. Bag A contains 2 red balls and 3 green balls. Two balls are chosen at random from the bag without replacement. Let X denote the number of red balls chosen. The following table shows the probability distribution for X.

X	0	1	2
$P(X = x)$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{1}{10}$

(a) Calculate E(X), the mean number of red balls chosen.

Bag B contains 4 red balls and 2 green balls. Two balls are chosen at random from bag B.

(b) (i) Draw a tree diagram to represent the above information, including the probability of each event.

(ii) Hence find the probability distribution for Y, where Y is the number of red balls chosen.

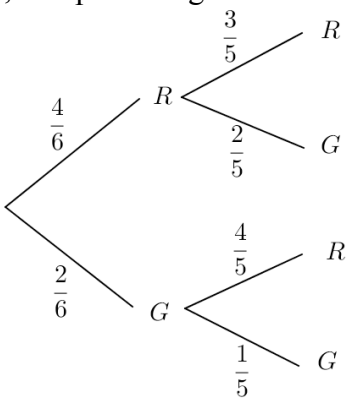
A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.

(c) Calculate the probability that two red balls are chosen.

(d) Given that two red balls are obtained, find the conditional probability that a 1 or 6 was rolled on the die.

(a) $E(X) = 0 \cdot \frac{3}{10} + 1 \cdot \frac{6}{10} + 2 \cdot \frac{1}{10} = \frac{0}{12} + \frac{6}{10} + \frac{2}{10} = \frac{8}{10}$

(b) (i) When it says that two balls are chosen, the presumption is that they are chosen together. That must mean there is no replacement. If there were, you wouldn't actually *have* two balls at the end. R is a red ball, G represents green.



(ii) Finding a probability distribution here means making a table like the one given initially.

P(green, then green) = $\frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30}$

P(red, then red) = $\frac{4}{6} \cdot \frac{3}{5} = \frac{12}{30}$

P(one of each color) = $1 - \frac{2}{30} - \frac{12}{30} = \frac{16}{30}$

Y	0	1	2
$P(Y = y)$	$\frac{2}{30}$	$\frac{16}{30}$	$\frac{12}{30}$

(c) P(2 reds) = P(bag A and 2 reds) + P(bag B and 2 reds)

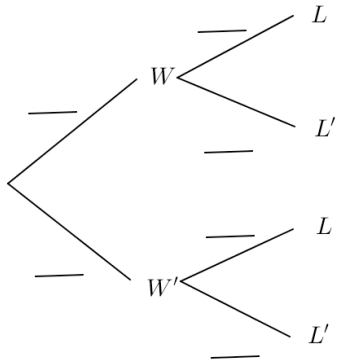
$$= \frac{2}{6} \cdot \frac{1}{10} + \frac{4}{6} \cdot \frac{12}{30} = \frac{2}{60} + \frac{48}{180} = \frac{1}{30} + \frac{8}{30} = \frac{9}{30} = \frac{3}{10}$$

(d) Getting a 1 or a 6 is identical to choosing from bag A.

$$\text{P(A} \mid \text{2 reds)} = \frac{\text{P}(A \cap \text{2 reds})}{\text{P(2 reds)}} = \frac{\frac{1}{3} \cdot \frac{1}{10}}{\frac{3}{10}} \frac{\frac{1}{30}}{\frac{3}{10}} = \frac{1}{30} \cdot \frac{10}{3} = \frac{1}{9}$$

14. Dumisani is a student at IB World College.
The probability that he will be woken by his alarm clock is
If he is woken by his alarm clock the probability he will be late for school is
If he is not woken by his alarm clock the probability he will be late for school is
Let W be the event "Dumisani is woken by his alarm clock".
Let L be the event "Dumisani is late for school".

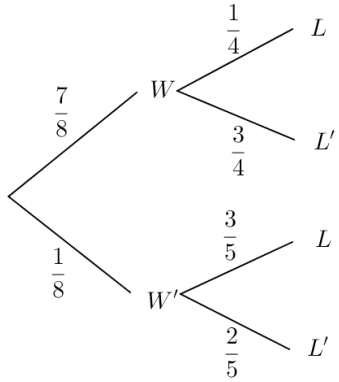
(a) Copy and complete the tree diagram below.



(b) Calculate the probability that Dumisani will be late for school.

(c) Given that Dumisani is late for school what is the probability that he was woken by his alarm clock?

(a) The "copy" instruction is because this question was from section B, where you write in a separate answer booklet.



(b) $\text{P}(L) = \frac{7}{8} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{3}{5} = \frac{47}{160}$

(c) $\text{P}(W \mid L) = \frac{\text{P}(W \cap L)}{\text{P}(L)} = \frac{\frac{7}{8} \cdot \frac{1}{4}}{\frac{47}{160}} = \frac{35}{47}$

15. The table below shows the subjects studied by 210 students at a college.

	Year 1	Year 2	Totals
History	50	35	85
Science	15	30	45
Art	45	35	80
Totals	110	100	210

(a) A student from the college is selected at random.
Let A be the event the student studies Art.
Let B be the event the student is in Year 2.
(i) Find P(A).
(ii) Find the probability that the student is a Year 2 Art student.
(iii) Are the events A and B independent? Justify your answer.

(b) Given that a History student is selected at random, calculate the probability that the student is in Year 1.

(c) Two students are selected at random from the college. Calculate the probability that one student is in Year 1, and the other in Year 2.

(a) (i) $\mathrm{P}(A) = \dfrac{n(A)}{n(U)} = \dfrac{80}{210} = \dfrac{8}{21}$

(ii) $\mathrm{P}(B \cap A) = \dfrac{n(B \cap A)}{n(U)} = \dfrac{35}{210} = \dfrac{1}{6}$

(iii) If A and B were independent, then it would be true that $\mathrm{P}(B \cap A) = \mathrm{P}(B) \cdot \mathrm{P}(A)$.

$\mathrm{P}(B) \cdot \mathrm{P}(A) = \dfrac{100}{210} \cdot \dfrac{8}{21} = \dfrac{80}{441} \neq \dfrac{1}{6}$, so A and B are not independent.

(b) $\mathrm{P}(B' \mid H) = \dfrac{n(B' \cap H)}{n(H)} = \dfrac{50}{85} = \dfrac{10}{17}$

(c) If we end up with two students, there must be no replacement. The first one could be year 1, or the second could.

$\mathrm{P}(\text{year 1, then year 2}) + \mathrm{P}(\text{year 2, then year 1}) = \dfrac{110}{210} \cdot \dfrac{100}{209} + \dfrac{100}{210} \cdot \dfrac{110}{209} = \dfrac{200}{399}$

16. In a bilingual school there is a class of 21 pupils. In this class, 15 of the pupils speak Spanish as their first language and 12 of these 15 pupils are Argentine. The other 6 pupils in the class speak English as their first language and 3 of these 6 pupils are Argentine.
A pupil is selected at random from the class and is found to be Argentine. Find the probability that the pupil speaks Spanish as his/her first language.

Either a Venn diagram or a table seems appropriate to organize the information. I'm using a table.

	Argentine	Other nationality	Totals
Spanish	12	**3**	15
English	3	**3**	6
Totals	**15**	**6**	21

$$\text{P(Spanish | Argentine)} = \frac{n(\text{Spanish and Argentine})}{n(\text{Argentine})} = \frac{12}{15} = \frac{4}{5}$$

17. In a game a player rolls a biased four-faced die. The probability of each possible score is shown below.

Score	1	2	3	4
Probability	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	x

(a) Find the value of x.
(b) Find E(X).
(c) The die is rolled twice. Find the probability of obtaining two scores of 3.

(a) $\frac{1}{5} + \frac{2}{5} + \frac{1}{10} + x = 1$

$x = \frac{3}{10}$

(b) $\text{E}(X) = \frac{1}{5} \cdot 1 + \frac{2}{5} \cdot 2 + \frac{1}{10} \cdot 3 + \frac{3}{10} \cdot 4 = \frac{5}{2}$

(c) Dice rolls are independent of each other.

$\frac{1}{10} \cdot \frac{1}{10} = \frac{1}{100}$

18. A biased die with four faces is used in a game. A player pays 10 counters to roll the die. The table below shows the possible scores on the die, the probability of each score and the number of counters the player receives in return for each score.

Score	1	2	3	4
Probability	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
Number of counters player receives	4	5	15	n

Find the value of n in order for the player to get an expected return of 9 counters per roll.

The expected value is the sum of the payoffs times the probability of each payoff. The "score" on the die is not part of that calculation.

$\frac{1}{2} \cdot 4 + \frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 15 + \frac{1}{10} \cdot n - 10 = 9$

$2 + 1 + 3 + \frac{1}{10}n = 19$

$\frac{1}{10}n = 13$

$n = 130$

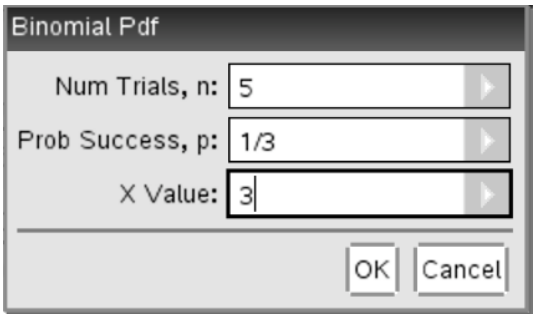
19. In a school, $\frac{1}{3}$ of the students travel to school by bus. Five students are chosen at random. Find the probability that exactly 3 of them travel to school by bus.

This is straight-up binomial probability. Let X represent the number of students that travel by bus.

Then $X \sim \text{B}\left(5, \frac{1}{3}\right)$.

$$\text{P}(X = 3) = \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = 10 \cdot \frac{1}{27} \cdot \frac{4}{9} = \frac{40}{243}$$

Or, equivalently (and much more easily), you can use the binomial PDF command on your calculator.



$\text{binomPdf}\left(5, \frac{1}{3}, 3\right)$ 0.164609

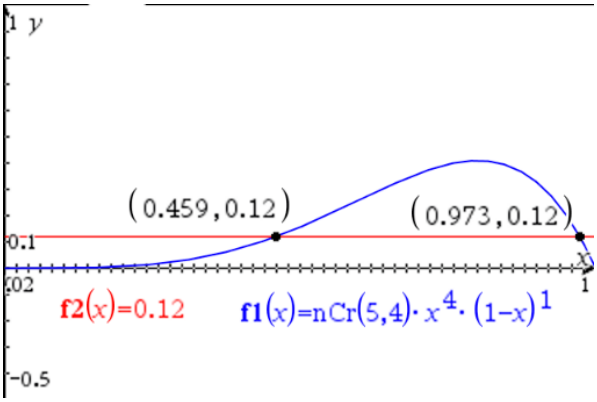
Those answers are the same.

20. X is a binomial random variable, where the number of trials is 5 and the probability of success of each trial is p. Find the values of p if $\text{P}(X = 4) = 0.12$.

The by-hand work in the previous problem may help you see how this is set up.

$$\binom{5}{4} p^4(1-p)^1 = 0.12$$

You would definitely have a calculator to help solve this, as the polynomial has both 5th and 4th degree terms. You might as well just graph it. On the IB exam, the nsolve command only finds one answer at a time. Because this is a 5th degree polynomial, there may be as many as 5 solutions. The only ones that make any sense would have to be valid probabilities, $0 \le p \le 1$. I will graph the two sides on this interval.



There turn out to be two solutions that are valid probabilities: $p \approx 0.459$ and $p \approx 0.973$.

21. When a boy plays a game at a fair, the probability that he wins a prize is 0.25. He plays the game 10 times. Let X denote the total number of prizes that he wins. Assuming that the games are independent, find

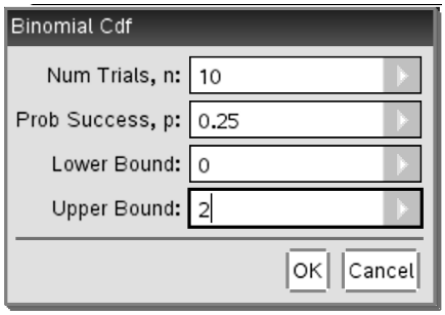
(a) E(X)

(b) P ($X \le 2$).

(a) Because the games are independent and there are repeated trials with exactly two outcomes (win and not win), this is again a binomial situation. It is especially easy to find the expected value of a binomial situation.

$E(X) = np = 10 \cdot 0.25 = 2.5$ prizes

(b) You can find this by adding up the probability that he wins 0, 1, or 2 prizes, but the binomial CDF command is made for this sort of thing.



binomCdf(10,0.25,0,2) 0.525593

So the probability is about 0.526. If you work it out by hand, you get an exact answer of $\binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} + \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^9 + \binom{10}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^8 = \frac{137\,781}{262\,144}$. Yeah, the first version was probably more useful.