Algebra, Functions and Equations
Worked solutions for practice problems

1. In an arithmetic sequence, the first term is 5 and the fourth term is 40. Find the second term.

$u_n = u_1 + (n-1)d$

$40 = 5 + (4-1)d$

$d = \dfrac{40-5}{3} = \dfrac{35}{3}$

$u_2 = 5 + \dfrac{35}{3} = \dfrac{50}{3}$

Note that, in general, alternative forms of numbers are fine if the question doesn't ask for a particular form. In this case, you could also give the answer as $16\frac{2}{3}$, $16.\bar{6}$, or 16.7 (correct to 3 significant figures).

2. An arithmetic series has five terms. The first term is 2 and the last term is 32. Find the sum of the series.

$S_n = \dfrac{n}{2}(u_1 + u_n)$

$S_5 = \dfrac{5}{2}(2+32) = \dfrac{5}{2} \cdot 34 = 85$

3. \$1000 is invested at the beginning of each year for 10 years.
The rate of interest is fixed at 7.5% per annum. Interest is compounded annually.
Calculate, giving your answers to the nearest dollar

(a) how much the first \$1000 is worth at the end of the ten years;

(b) the total value of the investments at the end of the ten years.

(a) You would get a calculator for this problem. This is a geometric sequence, with $r = 1.075$.

$u_n = u_1 \cdot r^{n-1}$

However, since the first \$1000 is put in *at the beginning* of the first year, that exponent will be 10 and not 9, as 10 years will have passed by the end of the tenth year.

$u_{10} = 1000 \cdot 1.075^{10} \approx 2061.03156... \approx \$2061$, to the nearest dollar

(b) This is more complicated. The first deposit earns interest for 10 years, the second for 9 years, the third for 8 years, and so on. That looks like this:

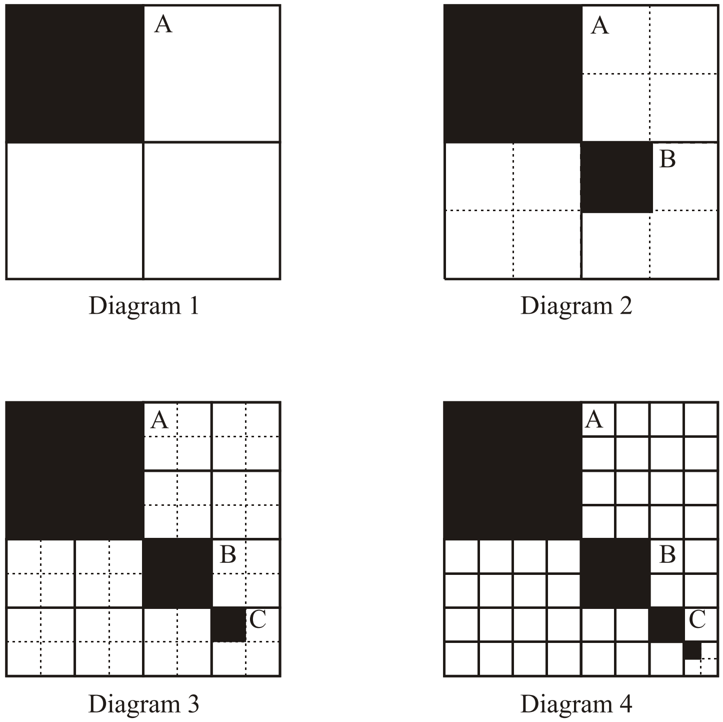
$1000 \cdot 1.075^{10} + 1000 \cdot 1.075^{9} + 1000 \cdot 1.075^{8} + \ldots + 1000 \cdot 1.075^{1}$

$1000(1.075^{10} + 1.075^{9} + 1.075^{8} + \ldots + 1.075^{1})$

I will use a geometric series formula for the part in parentheses, but it might be easier just to type the terms.

$1000 \cdot \dfrac{1.075(1-1.075^{10})}{1-1.075} \approx 15208.1190... \approx \$15208$, to the nearest dollar

4. **The diagrams below show the first four squares in a sequence of squares which are subdivided in half. The area of the shaded square A is $\frac{1}{4}$.**



(a) **(i)** **Find the area of square B and of square C.**

(ii) **Show that the areas of squares A, B and C are in geometric progression.**

(iii) **Write down the common ratio of the progression.**

(b) **(i)** **Find the total area shaded in diagram 2.**

(ii) **Find the total area shaded in the 8th diagram of this sequence. Give your answer correct to six significant figures.**

(c) **The dividing and shading process illustrated is continued indefinitely. Find the total area shaded.**

(a) (i) square B: $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

square C: $\frac{1}{16} \cdot \frac{1}{4} = \frac{1}{64}$

(ii) $\frac{\text{square B}}{\text{square A}} = \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{4}$

$\frac{\text{square C}}{\text{square B}} = \frac{\frac{1}{64}}{\frac{1}{16}} = \frac{1}{4}$

Since the ratios of successive terms are constant, the sequence is geometric.

(iii) $r = \frac{1}{4}$

(b) (i) $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$

(ii) $S_n = \dfrac{u_1(1 - r^n)}{1 - r}$

$$S_8 = \frac{\frac{1}{4}\left(1 - \left(\frac{1}{4}\right)^8\right)}{1 - \frac{1}{4}} = \frac{21845}{65536} \approx 0.333328 \text{ (to 6 s. f.)}$$

Notice that in this case, the exact answer will *not* get full credit, as the question specifically requires six significant figures. You would certainly be allowed a calculator for this.

(c) $S = \dfrac{u_1}{1 - r} = \dfrac{1/4}{1 - \frac{1}{4}} = \dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$

5. The following table shows four series of numbers. One of these series is geometric, one of the series is arithmetic and the other two are neither geometric nor arithmetic.

(a) Complete the table by stating the type of series that is shown.

Series		Type of series
(i)	$1 + 11 + 111 + 1111 + 11111 + \ldots$	
(ii)	$1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots$	
(iii)	$0.9 + 0.875 + 0.85 + 0.825 + 0.8 + \ldots$	
(iv)	$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \ldots$	

(b) The geometric series can be summed to infinity. Find this sum.

(a) (i) neither
(ii) geometric
(iii) arithmetic
(iv) neither

(b) $S = \dfrac{u_1}{1 - r} = \dfrac{1}{1 - \frac{3}{4}} = \dfrac{1}{\frac{1}{4}} = 4$

6. Let $\log_{10} P = x$, $\log_{10} Q = y$ and $\log_{10} R = z$. Express $\log_{10}\left(\dfrac{P}{QR^3}\right)^2$ in terms of x, y and z.

$$\log_{10}\left(\frac{P}{QR^3}\right)^2 = 2\log_{10}\left(\frac{P}{QR^3}\right) = 2\left(\log_{10} P - \log_{10}\left(QR^3\right)\right)$$

$$= 2\left(\log_{10} P - \left(\log_{10} Q + \log_{10}\left(R^3\right)\right)\right) = 2\left(\log_{10} P - \left(\log_{10} Q + 3\log_{10} R\right)\right)$$

$$= 2(x - (y + 3z)) = 2(x - y - 3z) = 2x - 2y - 6z$$

According to the markscheme either of those last two expressions would get the final mark, but the one with nested parentheses would not. You have to finish distributing that negative sign.

7. Given that $\log_5 x = y$, express each of the following in terms of y.

(a) $\log_5 x^2$

(b) $\log_5 \left(\frac{1}{x}\right)$

(c) $\log_{25} x$

(a) $\log_5 x^2 = 2\log_5 x = 2y$

(b) $\log_5 \left(\frac{1}{x}\right) = \log_5 1 - \log_5 x = 0 - y = -y$

(c) $\log_{25} x = \frac{\log_5 x}{\log_5 25} = \frac{y}{2}$

8. Given that $p = \log_a 5$, $q = \log_a 2$, express the following in terms of p and/or q.

(a) $\log_a 10$

(b) $\log_a 8$

(c) $\log_a 2.5$

(a) $\log_a 10 = \log_a(2 \cdot 5) = \log_a 2 + \log_a 5 = q + p$

(b) $\log_a 8 = \log_a \left(2^3\right) = 3\log_a 2 = 3q$

(c) $\log_a 2.5 = \log_a \frac{5}{2} = \log_a 5 - \log_a 2 = p - q$

9. Find the coefficient of x^3 in the expansion of $(2 - x)^5$.

The general formula for any term in the expansion of $(a + b)^n$ can be written as $\binom{n}{r} a^{n-r}b^r$. In this case, we need the exponent on b to be 3. Therefore $r = 3$.

$\binom{5}{3} 2^{5-3}(-x)^3 = 10 \cdot 4 \cdot -x^3 = -40x^3$

The coefficient is –40.

You might have a calculator for the binomial coefficient, or you might need to use either Pascal's triangle or the formula for $\binom{n}{r}$ in the formula booklet.

10. When the expression $(2 + ax)^{10}$ is expanded, the coefficient of the term in x^3 is 414 720. Find the value of a.

Again, the general formula for any term in the expansion of $(a + b)^n$ can be written as $\binom{n}{r} a^{n-r}b^r$. And again, we need the exponent on b to be 3. Therefore $r = 3$. Weird.

$\binom{10}{3} 2^{10-3}(ax)^3 = 120 \cdot 128 \cdot a^3x^3 = 15360a^3x^3$

The coefficient there is equal to 414 720:

$15360a^3 = 414\,720$

$a^3 = 27$

$a = 3$

11. Find the term in x^3 in the expansion of $\left(\frac{2}{3}x - 3\right)^8$.

This time in $\binom{n}{r} a^{n-r}b^r$, we need $n - r$ to be 3, so $r = 5$.

$$\binom{8}{5}\left(\frac{2}{3}x\right)^{8-5}(-3)^5 = 56 \cdot \frac{8}{27}x^3 \cdot -243 = -4032x^3$$

12. Let $f(x) = \sqrt{x+4}$, $x \geq -4$ and $g(x) = x^2, x \in \mathbb{R}$.

(a) Find $(g \circ f)(3)$.

(b) Find $f^{-1}(x)$.

(c) Write down the domain of f^{-1}.

(a) $g(f(3)) = g\left(\sqrt{7}\right) = \left(\sqrt{7}\right)^2 = 7$

(b) First switch x and y, then solve for y.

$x = \sqrt{y+4}$

$x^2 = y + 4$

$y = f^{-1}(x) = x^2 - 4$

(c) The domain of f^{-1} is the same as the range of f. Since f is a principal square root, its range is all nonnegative values; therefore the domain of f^{-1} is $x \geq 0$.

13. Show that $\frac{1}{\sqrt{n} + \sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0, n \in \mathbb{Z}$.

$$\frac{1}{\sqrt{n}+\sqrt{n+1}} \cdot 1 = \frac{1}{\sqrt{n}+\sqrt{n+1}} \cdot \frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}-\sqrt{n+1}} = \frac{\sqrt{n}-\sqrt{n+1}}{n-(n+1)} = \frac{\sqrt{n}-\sqrt{n+1}}{-1} = \sqrt{n+1}-\sqrt{n}$$

14. (a) Explain why any integer can be written in the form $4k$ or $4k + 1$ or $4k + 2$ or $4k + 3$, where $k \in \mathbb{Z}$.

(b) Hence prove that the square of any integer can be written in the form $4t$ or $4t + 1$, where $t \in \mathbb{Z}^+$.

(a) When any integer is divided by 4, it must have a remainder of 0, 1, 2, or 3. Therefore each integer can be written in the form $4k$ plus that remainder, i.e. as $4k$, $4k + 1$, $4k + 2$, or $4k + 3$.

(b) Checking each of the squares from (a):

$(4k)^2 = 16k^2 = 4 \cdot 4k^2 = 4t$, where $t = 4k^2$.

$(4k+1)^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1 = 4t + 1$, where $t = 4k^2 + 2k$.

$(4k+2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1) = 4t$, where $t = 4k^2 + 4k + 1$.

$(4k+3)^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1 = 4t + 1$, where $t = 4k^2 + 6k + 2$.

Therefore any integer can be written in the for $4t$ or $4t + 1$ where $t \in \mathbb{Z}$.

15. **(a)** **Prove that** $\log_a b = \log_{\sqrt{a}} \sqrt{b}$ **where** $a, b \in \mathbb{R}^+$.

(b) **Evaluate** $\dfrac{\log_9 36 - \log_3 2}{1 + \log_4 2}$.

(a) Let $\log_a b = x$. Then $b = a^x$. Taking the square root of both sides gives $\sqrt{b} = \sqrt{a^x} = \left(\sqrt{a}\right)^x$. Changing back to logarithms gives $\log_{\sqrt{a}} \sqrt{b} = x$. Therefore $\log_a b = \log_{\sqrt{a}} \sqrt{b}$.

(b) Using the result from (a) to rewrite the first term in the numerator gives

$$\frac{\log_3 6 - \log_3 2}{1 + \log_4 2} = \frac{\log_3 \frac{6}{2}}{1 + \frac{1}{2}} = \frac{\log_3 3}{\frac{3}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

16. **Let** $\log_a x = m$ **and** $\log_a y = n$.

(a) **In terms of *a*, *m*, and/or *n* write down an expression for**

(i) x

(ii) y.

(b) **By determining an expression for** $\dfrac{x}{y}$, **prove that** $\log_a x - \log_a y = \log_a \dfrac{x}{y}$.

(a) (i) $x = a^m$

(ii) $y = a^n$

(b) $\dfrac{x}{y} = \dfrac{a^m}{a^n}$, so $\log_a \dfrac{x}{y} = \log_a \dfrac{a^m}{a^n} = \log_a a^{m-n} = m - n = \log_a x - \log_a y$.

17. **Consider the finite geometric sequence 4, 8, 16, 32.**

(a) **Write down**

(i) **the common ratio;**

(ii) **the sum of the sequence.**

(b) **Show that there are no other geometric sequences with 4 terms and first term equal to 4 whose sum is equal to the value from (a) part (ii).**

(a) (i) $r = 2$

(ii) $4 + 8 + 16 + 32 = 60$

(b) $4 + 4r + 4r^2 + 4r^3 = 60$

$1 + r + r^2 + r^3 = 15$

$r^3 + r^2 + r + 14 = 0$

We know that $r = 2$ is a solution of the equation from (a), so long division (or synthetic division, or factoring) tells us that $r^3 + r^2 + r + 14 = (r - 2)(r^2 + 3r + 7) = 0$.

The discriminant of $r^2 + 3r + 7 = 3^2 - 4 \cdot 1 \cdot 7 = -19$, indicating that the second factor will only have imaginary zeros, and there is no other real value of r that produces this sum.

18. **(a)** **Prove that** $\log_a b = \dfrac{1}{\log_b a}$ **for** $a, b > 0$.

Let $f(x) = 4^x$**. The value of** c **is such that** $f(c) = 7$**.**

(b) **Find the value of** c**.**

(c) **For the graph of** $y = f(x)$**, write down**

(i) **the coordinates of the y-intercept;**

(ii) **the equation of the asymptote.**

The graph of $y = f(x)$ **is translated by** c **units to the left resulting in the graph of** $y = g(x)$**.**

(d) **Write down the function** $g(x)$**.**

(e) **Find a different transformation that can be applied to the graph of** $y = f(x)$ **to produce the graph of** $y = g(x)$**.**

A stretch by a factor of $\log_a b$ **is applied to the graph of** $y = f(x)$ **to produce the graph of** $y = 7^x$**.**

(f) **State whether the stretch is horizontal or vertical and find the values of** a **and** b**.**

(a) Using the change of base formula, $\log_a b = \dfrac{\log_b b}{\log_b a} = \dfrac{1}{\log_b a}$.

(b) $f(c) = 4^c = 7$, so $c = \log_4 7$.

(c) (i) $f(0) = 4^0 = 1$, so the y-intercept is (0, 1).

(ii) $y = 0$, since $\lim\limits_{x \to \infty} 4^x = 0$.

(d) $g(x) = 4^{x+c} = 4^{x+\log_4 7}$

(e) $g(x) = 4^{x+\log_4 7} = 4^x \cdot 4^{\log_4 7} = 4^x \cdot 7$, which is a vertical stretch by a factor of 7.

(f) $7^x = 4^{\log_4 7^x} = 4^{x \log_4 7}$, which gives a horizontal stretch with a scale factor of $\dfrac{1}{\log_4 7}$. Using the result of part (a), $\dfrac{1}{\log_4 7} = \log_7 4$, so $a = 7$ and $b = 4$.

19. **Let** $g(x) = 3x - 2$, $h(x) = \dfrac{5x}{x-4}$, $x \neq 4$.

(a) **Find an expression for** $(h \circ g)(x)$**. Simplify your answer.**

(b) **Solve the equation** $(h \circ g)(x) = 0$**.**

(a) $h(g(x)) = \dfrac{5 \cdot g(x)}{g(x) - 4} = \dfrac{5(3x-2)}{(3x-2)-4} = \dfrac{15x - 10}{3x - 6}$

Note that you do not get the last answer mark unless you work out the denominator.

(b) $\dfrac{15x - 10}{3x - 6} = 0$

$(3x - 6)\left(\dfrac{15x - 10}{3x - 6}\right) = 0 \cdot (3x - 6)$

$15x - 10 = 0$

$x = \dfrac{10}{15} = \dfrac{2}{3}$

20. Consider the functions $f(x) = 2x$ and $g(x) = \dfrac{1}{x-3}, x \neq 3$.

(a) Calculate $(f \circ g)(4)$.

(b) Find $g^{-1}(x)$.

(c) Write down the domain of g^{-1}.

(a) $(f \circ g)(4) = f\left(\dfrac{1}{4-3}\right) = f(1) = 2 \cdot 1 = 2$

(b) First switch x and y, then solve for y.

$x = \dfrac{1}{y-3}$

I like the Flip That Sucker Over method here; if two fractions are equal to each other, then their reciprocals are also equal.

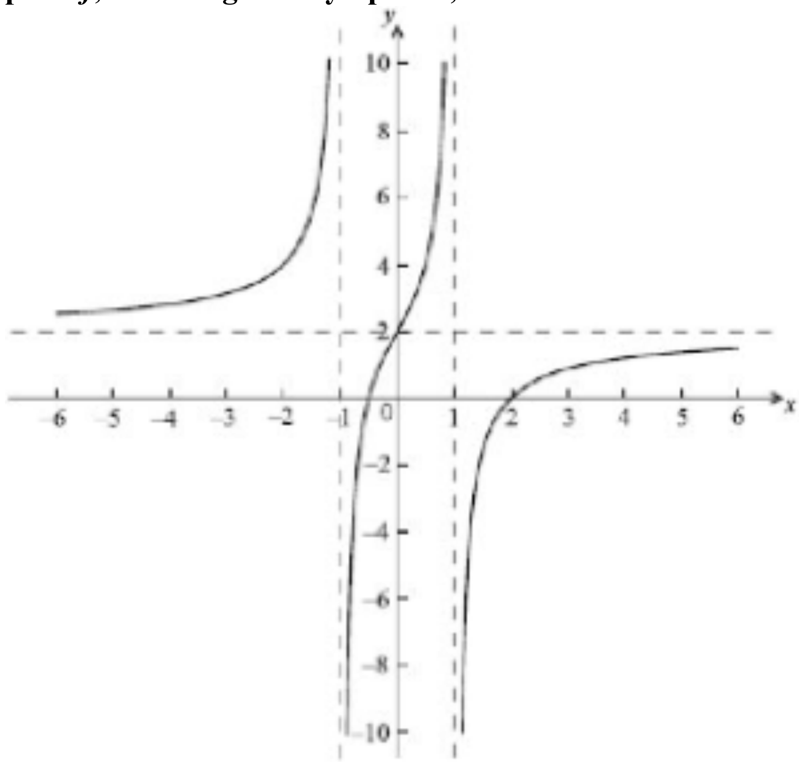
$\dfrac{1}{x} = y - 3$

$y = g^{-1}(x) = \dfrac{1}{x} + 3$

(c) The domain of g^{-1} is the range of g. That fraction for g can be anything except 0, so the domain of g^{-1} is $x \neq 0$. That makes sense in the context of the answer to part (b), too.

21. Let $f(x) = p - \dfrac{3x}{x^2 - q^2}$, where $p, q \in \mathbb{R}^+$.

Part of the graph of *f*, including the asymptotes, is shown below.



(a) The equations of the asymptotes are $x = 1, x = -1, y = 2$. Write down the value of

(i) p;

(ii) q.

(b) Let R be the region bounded by the graph of *f*, the *x*-axis, and the *y*-axis.

(i) Find the negative *x*-intercept of *f*.

(ii) Hence find the volume obtained when *R* is revolved through 360° about the *x*-axis.

(c) (i) Show that $f'(x) = \dfrac{3(x^2+1)}{(x^2-1)^2}$.

(ii) Hence, show that there are no maximum or minimum points on the graph of *f*.

(d) Let $g(x) = f'(x)$. Let *A* be the area of the region enclosed by the graph of *g* and the *x*-axis, between $x = 0$ and $x = a$, where $a > 0$. Given that $A = 2$, find the value of *a*.

(a) (i) $p = 2$, since this is the limit as $x \to \infty$ of $f(x)$.

(ii) $q = 1$, because that makes the denominator factor to $(x+1)(x-1)$.

(b) (i) $2 - \dfrac{3x}{x^2-1} = 0$

$2 = \dfrac{3x}{x^2-1}$

$2x^2 - 2 = 3x$

$2x^2 - 3x - 2 = 0$

$(2x+1)(x-2) = 0$

$x = -\dfrac{1}{2}, x = 2$

So the negative *x*-intercept is $\left(-\dfrac{1}{2}, 0\right)$.

(ii) $\displaystyle\int_{-\frac{1}{2}}^{0} \pi\left(f(x)\right)^2 dx = \int_{-\frac{1}{2}}^{0} \pi\left(2 - \frac{3x}{x^2-1}\right)^2 dx \approx 2.51965 \approx 2.52$ (3 s.f.)

Note that you *are* allowed a calculator on this problem, so to attempt to do that integral by hand is a waste of time and effort, and also that you are *not* allowed a CAS on the IB exam, so if your answer has a natural logarithm in it, you've used technology that is prohibited.

(c) (i) $\dfrac{d}{dx}\left(2 - \dfrac{3x}{x^2-1}\right) = 0 - \dfrac{(x^2-1)\cdot 3 - 3x(2x)}{(x^2-1)^2} = -\dfrac{3x^2-3-6x^2}{(x^2-1)^2}$

$= -\dfrac{-3x^2-3}{(x^2-1)^2} = \dfrac{3x^2+3}{(x^2-1)^2} = \dfrac{3(x^2+1)}{(x^2-1)^2}$

(ii) Maximum and minimum points can only occur at critical points and endpoints. There are no endpoints for this function. We see that $f'(x)$ is undefined only when $x^2 - 1 = 0$, which means *f* would also be undefined. There are no points there, so there cannot be a maximum or minimum point there, either. Finally, $f'(x) = 0$ would mean that $x^2 + 1 = 0$; that equation has only imaginary solutions. Therefore *f* has no maximum or minimum points.

(d) $\displaystyle\int_0^a g(x)\,dx = \int_0^a f'(x)\,dx = 2$

Since we know the antiderivative of this function, we can evaluate the definite integral.

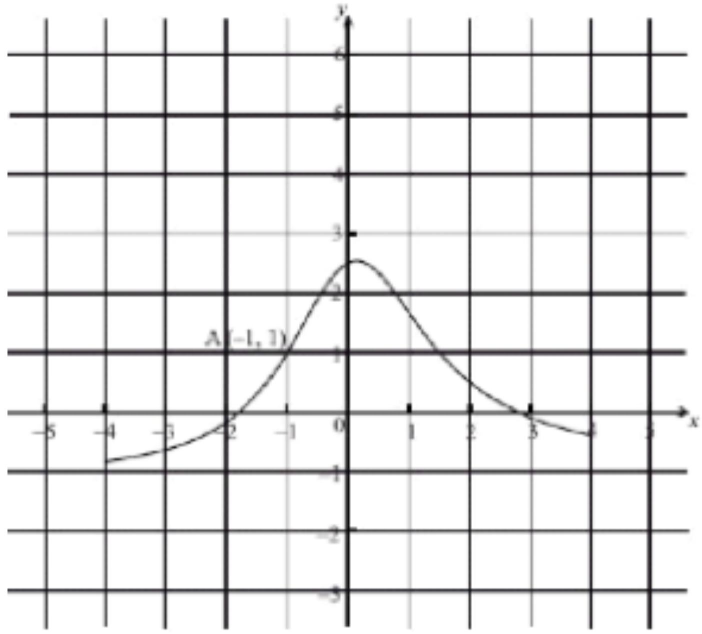
$\left[2 - \dfrac{3x}{x^2-1}\right]_0^a = \left(2 - \dfrac{3a}{a^2-1}\right) - \left(2 - \dfrac{3\cdot 0}{0^2-1}\right) = -\dfrac{3a}{a^2-1}$

And then we can solve the resulting equation.

$-\dfrac{3a}{a^2-1} = 2$

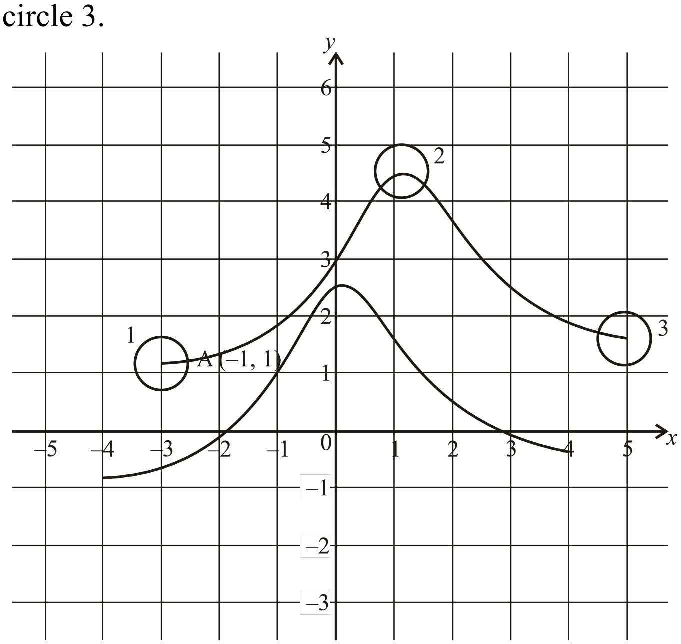
$-3a = 2a^2 - 2$
$0 = 2a^2 + 3a - 2$
$0 = (2a - 1)(a + 2)$
Therefore $a = \dfrac{1}{2}$ or $a = -2$. But we are given that $a > 0$, so the conclusion is that $a = \dfrac{1}{2}$.

22. The graph of a function *f* is shown in the diagram below. The point A (–1, 1) is on the graph, and $y = -1$ is a horizontal asymptote.



(a) Let $g(x) = f(x - 1) + 2$. On the diagram, sketch the graph of *g*.
(b) Write down the equation of the horizontal asymptote of *g*.
(c) Let A′ be the point on the graph of *g* corresponding to point A. Write down the coordinates of A′.

(a) One mark for the left endpoint in circle 1, one for the maximum in circle 2, and one for the right endpoint in circle 3.



(b) $y = 1$, translated up two units from the original horizontal asymptote. Note that this must be an equation of a line; just a number will not get the mark.

(c) Point A will have moved right one unit and up two units, so A′ is at (0, 3).

23. The function f is defined as $f(x) = (2x + 1)e^{-x}$, $0 \le x \le 3$. The point P(0, 1) lies on the graph of $f(x)$, and there is a maximum point at Q.

(a) Sketch the graph of $y = f(x)$, labelling the points P and Q.

(b) (i) Show that $f'(x) = (1 - 2x)e^{-x}$.

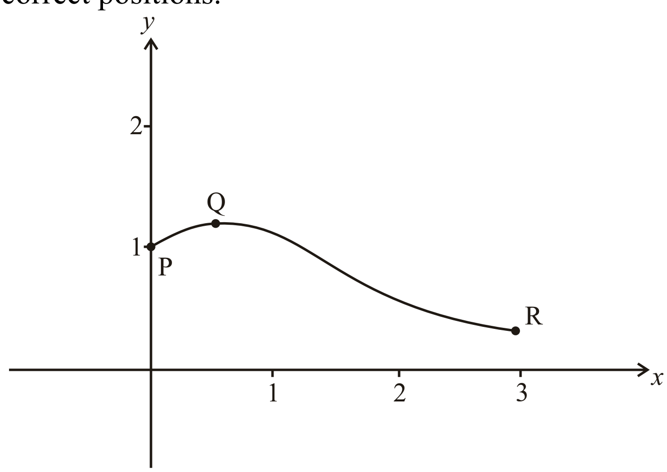
(ii) Find the *exact* coordinates of Q.

(c) The equation $f(x) = k$, where $k \in \mathbb{R}$, has two solutions. Write down the range of values of k.

(d) Given that $f''(x) = e^{-x}(-3 + 2x)$, show that the curve of f has only one point of inflexion.

(e) Let R be the point on the curve of f with x-coordinate 3. Find the area of the region enclosed by the curve and the line (PR).

(a) The marks are A1 for shape, A1 for correct domain, and A1 for labeling both points P and Q in approximately correct positions.



(b) (i) $f'(x) = (2x + 1) \cdot e^{-x} \cdot -1 + e^{-x} \cdot 2 = e^{-x}(-2x - 1 + 2) = (1 - 2x)e^{-x}$

(ii) At Q, $f'(x) = 0$.

$(1 - 2x)e^{-x} = 0 \Rightarrow 1 - 2x = 0$ or $e^{-x} = 0$

But $e^{-x} \neq 0$.

Therefore at Q, $x = \frac{1}{2}$.

$f\left(\frac{1}{2}\right) = \left(2 \cdot \frac{1}{2} + 1\right) \cdot e^{-\frac{1}{2}} = 2e^{-\frac{1}{2}}$

$Q\left(\frac{1}{2}, 2e^{-\frac{1}{2}}\right)$

(c) Based on the graph of f in part (a), the horizontal line $y = k$ can intersect the graph of f twice if $1 \le k < 2e^{-\frac{1}{2}}$.

(d) The graph of f will have a point of inflection (or, you know, inflexion) where f'' changes signs. Since $e^{-x} > 0$ for all x, this sign change can only happen when $-3 + 2x = 0$, and that has only one solution, $x = \frac{3}{2}$.

(e) The line (PR) has a y-intercept of 1 and a slope of $\dfrac{7e^{-3}-1}{3-0}=\dfrac{7e^{-3}-1}{3}$. Therefore its equation is $y=\dfrac{7e^{-3}-1}{3}x+1$. The area is given by the integral

$$\int_0^3\left((2x+1)e^{-x}-\left(\frac{7e^{-3}-1}{3}x+1\right)\right)dx\approx 0.529.$$

Again, you should pick up a calculator once you have written this integral. It would be fine if the slope were given as a decimal in the equation for line (PQ).

24. Consider the functions f and g where $f(x)=3x-5$ and $g(x)=x-2$.

(a) Find the inverse function, f^{-1}.

(b) Given that $g^{-1}(x)=x+2$, find $(g^{-1}\circ f)(x)$.

(c) Given also that $(f^{-1}\circ g)(x)=\dfrac{x+3}{3}$, solve $(f^{-1}\circ g)(x)=(g^{-1}\circ f)(x)$.

Let $h(x)=\dfrac{f(x)}{g(x)}$, $x\neq 2$.

(d) (i) *Sketch* the graph of h for $-3\le x\le 7$ and $-2\le y\le 8$, including any asymptotes.

(ii) Write down the *equations* of the asymptotes.

(e) The expression $\dfrac{3x-5}{x-3}$ may also be written as $3+\dfrac{1}{x-2}$. Use this to answer the following.

(i) Find $\int h(x)\,dx$.

(ii) *Hence*, calculate the *exact* value of $\int_3^5 h(x)\,dx$.

(f) On your sketch, shade the region whose area is represented by $\int_3^5 h(x)\,dx$.

(a) $x=3y-5$

$x+5=3y$

$y=f^{-1}(x)=\dfrac{x+5}{3}$

(b) $(g^{-1}\circ f)(x)=f(x)+2=3x-5+2=3x-3$

(c) $\dfrac{x+3}{3}=3x-3$

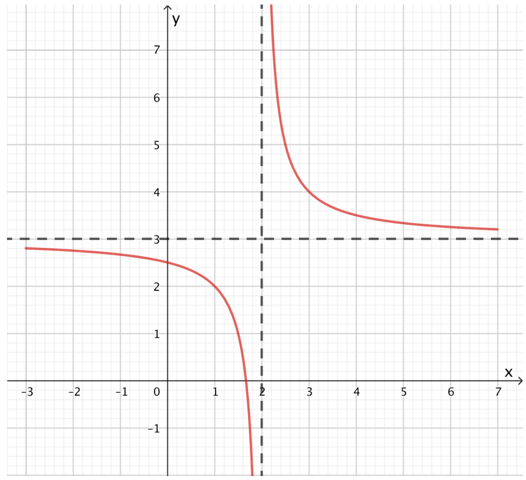
$x+3=9x-9$

$12=8x$

$x=\dfrac{12}{8}=\dfrac{3}{2}=1.5$

Any of the three values on the last line gets the last answer mark.

(d) (i) The first A1 is for correct domain and range, the second for two branches with approximately the correct shape, and the third for both asymptotes.

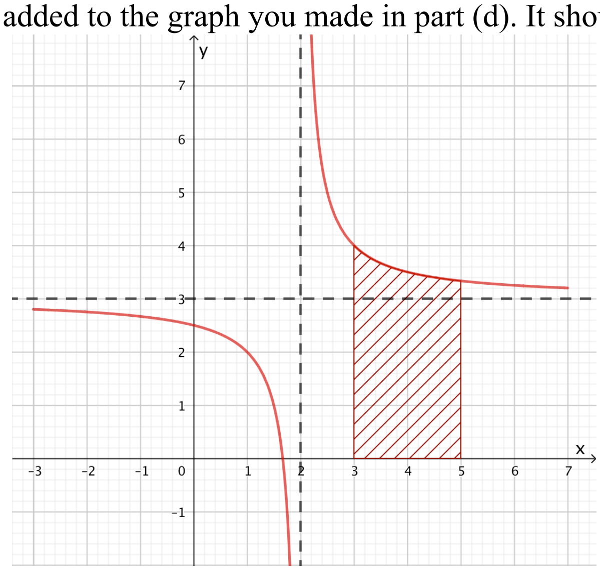


(ii) $x = 2, y = 3$

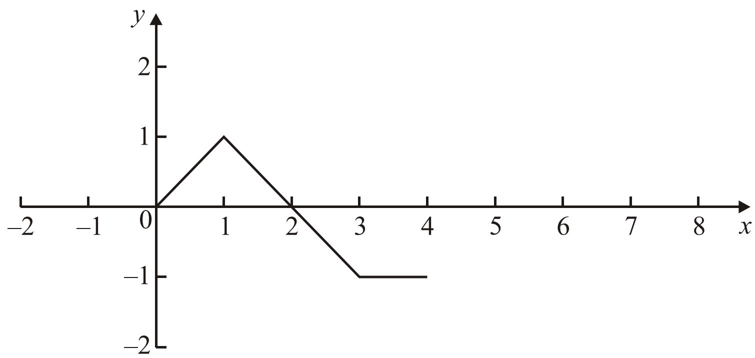
(e) (i) $\int \left(3 + \frac{1}{x-2}\right) dx = 3x + \ln|x-2| + C$

(ii) $\left[3x + \ln|x-2|\right]_3^5 = (3 \cdot 5 + \ln 3) - (3 \cdot 3 + \ln 1) = 6 + \ln 3$

(f) This shading would be added to the graph you made in part (d). It should *not* be a new graph.

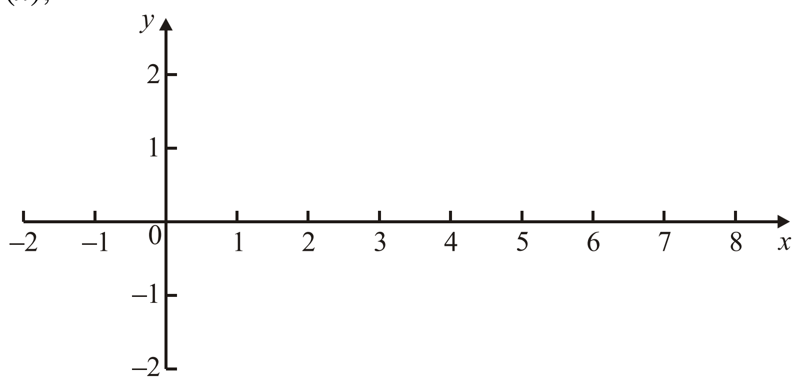


25. The graph of $y = f(x)$ is shown in the diagram.

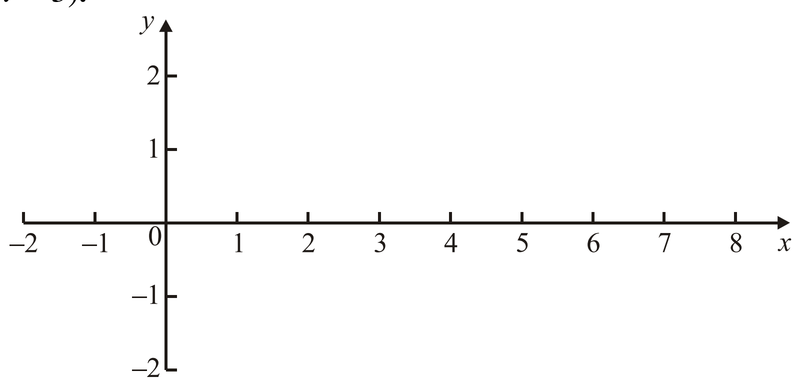


(a) On each of the following diagrams draw the required graph,

(i) $y = 2f(x)$;

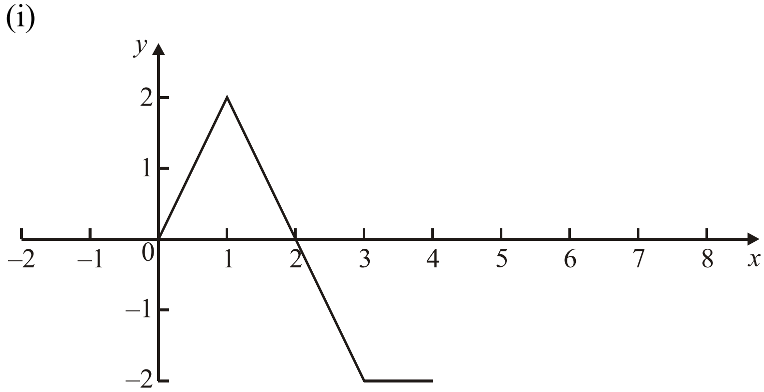


(ii) $y = f(x - 3)$.

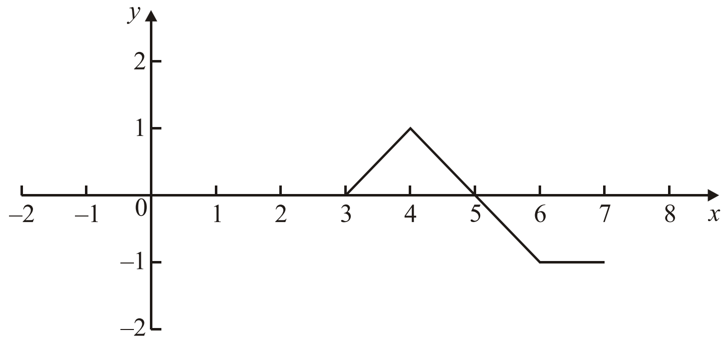


(b) The point A (3, –1) is on the graph of *f*. The point A′ is the corresponding point on the graph of $y = -f(x) + 1$. Find the coordinates of A′.

(a) (i)



(ii)



(b) The point is reflected in the x-axis and then translated up one unit. So $(3, -1)$ goes to $(3,1)$ and then to $(3, 2)$.

26. The area A km^2 affected by a forest fire at time t hours is given by $A = A_0 e^{kt}$.
When $t = 5$, the area affected is 1 km^2 and the rate of change of the area is $0.2\,\text{km}^2\,\text{h}^{-1}$.

(a) Show that $k = 0.2$.

(b) Given that $A_0 = \dfrac{1}{e}$, find the value of t when $100\,\text{km}^2$ are affected.

(a) Because 1 km^2 is affected at $t = 5$, $1 = A_0 e^{k \cdot 5}$.

The rate of change of the area is $\dfrac{d}{dx}\left[A_0 e^{kt}\right] = A_0 e^{kt} \cdot k$ in general. At $t = 5$, this becomes $\dfrac{d}{dx}\left[A_0 e^{kt}\right]\Big|_{t=5} = A_0 e^{k \cdot 5} \cdot k = 0.2$. Since we know that $A_0 e^{k \cdot 5} = 1$, substituting gives $1 \cdot k = k = 0.2$.

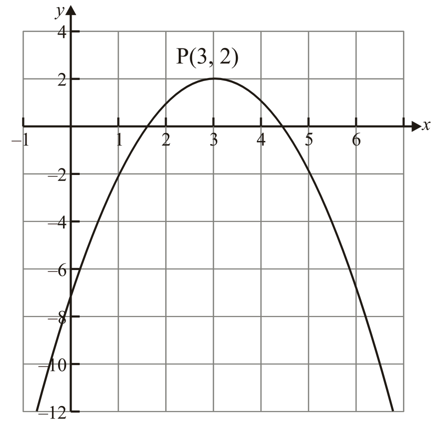
(b) With this value of A_0, we now have that $A = \dfrac{1}{e} \cdot e^{0.2t}$.

$100 = \dfrac{1}{e} \cdot e^{0.2t}$

$100 = e^{-1} \cdot e^{0.2t} = e^{-1+0.2t}$

Then $\ln 100 = -1 + 0.2t$, and $t = \dfrac{\ln 100 + 1}{0.2} = 5(\ln 100 + 1) \approx 28.0$ hours.

27. The function $f(x)$ is defined as $f(x) = -(x - h)^2 + k$. The diagram below shows part of the graph of $f(x)$. The maximum point on the curve is P (3, 2).



(a) Write down the value of

(i) h;

(ii) k.

(b) Show that $f(x)$ can be written as $f(x) = -x^2 + 6x - 7$.

(c) Find $f'(x)$.

The point Q lies on the curve and has coordinates (4, 1). A straight line L, through Q, is perpendicular to the tangent at Q.

(d) (i) Calculate the gradient of L.

(ii) Find the equation of L.

(iii) The line L intersects the curve again at R. Find the x-coordinate of R.

(a) (i) $h = 3$

(ii) $k = 2$

(b) $f(x) = -(x - 3)^2 + 2 = -(x^2 - 6x + 9) + 2$

$= -x^2 + 6x - 9 + 2 = -x^2 + 6x - 7$

(c) Using the power rule on part (b), $f'(x) = -2x + 6$.

(d) (i) $f'(4) = -2 \cdot 4 + 6 = -8 + 6 = -2$

The perpendicular to the tangent will therefore have a slope of $-\left(\frac{1}{-2}\right) = \frac{1}{2}$.

(ii) $y - 1 = \frac{1}{2}(x - 4)$

Slope-intercept form would also be fine here, as, presumably, would equivalent forms.

(iii) First rewriting the line to solve for y gives $y = \frac{1}{2}(x - 4) + 1$. Then setting the line and parabolic equations equal gives an equation to solve.

$\frac{1}{2}(x - 4) + 1 = -x^2 + 6x - 7$

$(x - 4) + 2 = -2x^2 + 12x - 14$

$2x^2 - 11x + 12 = 0$

$(2x - 3)(x - 4) = 0$

$x = \frac{3}{2}, x = 4$

As $x = 4$ corresponds to the coordinates given for point Q, the x-coordinate of R is $\frac{3}{2}$.

28. The population of a city at the end of 1972 was 250 000. The population increases by 1.3% per year.

(a) Write down the population at the end of 1973.

(b) Find the population at the end of 2002.

(a) $250\,000 \cdot 1.013 = 253\,250$ people (exactly, or 253 000 to 3 s.f.)

(b) Since 2002 is 30 years after 1972 there would be 30 years worth of population growth.

$250\,000 \cdot 1.013^{30} \approx 368\,318.361...$

To three significant figures, this is 368 000 people.