Geometry and Trigonometry
Worked solutions for practice problems

1. A metal sphere has a radius of 12.7 cm.

(a) Find the volume of the sphere, expressing your answer in the form $a \times 10^k, 1 \leq a < 10$ and $k \in \mathbb{Z}$.

(b) The sphere is to be melted down and remoulded into the shape of a cone with a height of 14.8 cm. Find the radius of the base of the cone, correct to 2 significant figures.

(a) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 12.7^3 \approx 8580.24 \approx 8.58 \times 10^3$ cm³

(b) Now this volume is equal to the volume of a cone.

$8580.24 = \frac{1}{3}\pi r^2 h$

$8580.24 = \frac{1}{3}\pi r^2 \cdot 14.8$

$r \approx 23.529 \approx 24$ cm to two s. f.

2. A cylinder with radius *r* and height *h* is shown in the following diagram.
The sum of *r* and *h* for this cylinder is 12 cm.

(a) Write down an equation for the area, *A*, of the curved surface in terms of *r*.

(b) Find $\frac{dA}{dr}$.

(c) Find the value of *r* when the area of the curved surface is maximized.

(a) Since $r + h = 12$, $h = 12 - r$. Then $A = 2\pi r h = 2\pi r(12 - r)$.

(b) $A = 2\pi r(12 - r) = 24\pi r - 2\pi r^2$

$\frac{dA}{dr} = 24\pi - 4\pi r$

(c) The maximum volume happens when that derivative is zero.

$\frac{dA}{dr} = 24\pi - 4\pi r = 0 \implies r = \frac{24\pi}{4\pi} = 6$ cm

3. A factory packages coconut water in cone-shaped containers with a base radius of 5.2 cm and a height of 13 cm.
The factory designers are currently investigating whether a cone-shaped container can be replaced with a cylinder-shaped container with the same radius and the same total surface area.

(a) Find the volume of one cone-shaped container.
(b) Find the slant height of the cone-shaped container.
(c) Show that the total surface area of the cone-shaped container is 314 cm², correct to three significant figures.
(d) Find the height *h*, of this cylinder-shaped container.
(e) The factory director wants to increase the volume of coconut water sold per container. State whether or not they should replace the cone-shaped containers with cylinder-shaped containers. Justify your conclusion.

(a) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 5.2^2 \cdot 13 \approx 368.110 \approx 368$ cm³

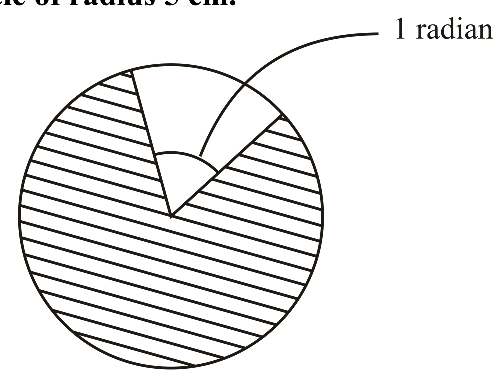
(b) The slant height is the hypotenuse of a right triangle with legs as the radius and height of the cone.
$l^2 = 5.2^2 + 13^2 \implies l = \sqrt{5.2^2 + 13^2} \approx 14.0014 \approx 14.0$ cm

(c) The total surface area is the sum of the curved (lateral) surface area and the area of the circular base.
$\pi r l + \pi r^2 = \pi(5.2)(14.0) + \pi(5.2)^2 \approx 313.679 \approx 314$ cm²

(d) The surface area of a cylinder is the sum of the curved surface area and the area of the two circular bases.
$2\pi r h + 2\pi r^2 = 2\pi(5.2)^2 h + 2\pi(5.2)^2 = 314.679$
Solving for h with a calculator gives $h \approx 4.41051 \approx 4.41$ cm.

(e) The volume of the cylindrical container is $\pi r^2 h = \pi(5.2)^2(4.41) \approx 374.666 \approx 375$, which is greater than the volume of the conical package. Yes, they should replace the cones with cylinders to be able to include more coconut water with the same surface area in packaging.

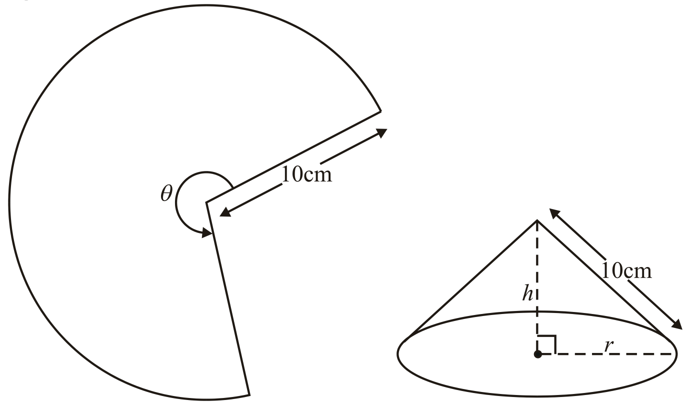
4. The diagram shows a circle of radius 5 cm.



Find the perimeter of the shaded region.

The perimeter consists of the major arc and two radii. There is a formula for arc length, which uses the angle in radians. The angle for the major arc is $2\pi - 1$.
Perimeter = $r\theta + 2r = 5(2\pi - 1) + 2 \cdot 5 = 10\pi - 5 + 10 = 10\pi + 5$ cm

5. The diagrams show a circular sector of radius 10 cm and angle θ radians which is formed into a cone of slant height 10 cm. The vertical height h of the cone is equal to the radius r of its base. Find the angle θ radians.



The key thing to understand here is the relationship between the two diagrams. That sector on the left *is* the lateral surface (everything but the base) of the cone on the right. That also means that the arc length of the sector is the same as the circumference of the base of the cone.

Arc length = 10θ

Circumference = $2\pi r$

Since the radius and the height of the cone are the same, that must be an isosceles right triangle, and the length of each leg is its hypotenuse divided by $\sqrt{2}$.

$r = \dfrac{10}{\sqrt{2}} = 5\sqrt{2}$

Therefore $10\theta = 2 \cdot \pi \cdot 5\sqrt{2}$, and $\theta = \pi\sqrt{2}$.

6. (a) Express $2\cos^2 x + \sin x$ in terms of sin *x* only.

(b) Solve the equation $2\cos^2 x + \sin x = 2$ for *x* in the interval $0 \le x \le \pi$, giving your answers exactly.

(a) $2\cos^2 x + \sin x = 2(1 - \sin^2 x) + \sin x = 2 - 2\sin^2 x + \sin x$

(b) $2 - 2\sin^2 x + \sin x = 2$

$-2\sin^2 x + \sin x = 0$

$-\sin x(2\sin x - 1) = 0$

$\sin x = 0$ or $\sin x = \dfrac{1}{2}$

$x = 0, \pi, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

7. **Solve the equation 3 cos x = 5 sin x, for x in the interval 0° ≤ x ≤ 360°, giving your answers to the nearest degree.**

$$\frac{3\cos x}{5\cos x} = \frac{5\sin x}{5\cos x}$$

$$\frac{3}{5} = \frac{\sin x}{\cos x} = \tan x$$

$$x \approx 31^\circ, 211^\circ$$

The second answer is in quadrant III, with the same reference angle. Alternatively, a graph would be a good method, but you would either have to graph in degrees (which is unusual) or convert the radian answers you get from the graph to degrees.

8. **Given that** $\sin x = \frac{1}{3}$**, where x is an acute angle, find the exact value of**

(a) $\cos x$**;**

(b) $\cos 2x$**.**

(a) $\sin^2 x + \cos^2 x = 1$

$$\left(\frac{1}{3}\right)^2 + \cos^2 x = 1$$

$$\cos^2 x = \frac{8}{9}$$

$$\cos x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

Either of the last two values are final; $\sqrt{9}$ is an integer and probably should be simplified. The positive square root was used because acute angles have positive cosine values.

(b) $\cos 2x = 1 - 2\sin^2 x$

$$\cos 2x = 1 - 2\left(\frac{1}{3}\right)^2 = 1 - \frac{2}{9} = \frac{7}{9}$$

9. **Consider the trigonometric equation** $2\sin^2 x = 1 + \cos x$**.**

(a) Write this equation in the form $f(x) = 0$**, where** $f(x) = a\cos^2 x + b\cos x + c$**, and** $a, b, c \in \mathbb{Z}$**.**

(b) Factorize $f(x)$**.**

(c) Solve $f(x) = 0$ **for 0° ≤ x ≤ 360°.**

(a) $2(1 - \cos^2 x) = 1 + \cos x$

$2 - 2\cos^2 x = 1 + \cos x$

$2\cos^2 x + \cos x - 1 = 0$

(b) $2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1)$

(c) $(2\cos x - 1)(\cos x + 1) = 0$

$$\cos x = \frac{1}{2} \text{ or } \cos x = -1$$

$$x = 60^\circ, 300^\circ, 180^\circ$$

10. (a) Factorize the expression $3\sin^2 x - 11\sin x + 6$.

(b) Consider the equation $3\sin^2 x - 11\sin x + 6 = 0$.

(i) Find the two values of sin *x* which satisfy this equation.

(ii) Solve the equation, for 0° ≤ *x* ≤ 180°.

(a) $3\sin^2 x - 11\sin x + 6 = (3\sin x - 2)(\sin x - 3)$

(b) (i) $(3\sin x - 2)(\sin x - 3) = 0$

$\sin x = \frac{2}{3}$ or $\sin x = 3$

Despite the fact that there are no values of sin *x* that are greater than 1, both values are needed here.

(ii) $\sin x = \frac{2}{3}$ gives $x \approx 41.8°, 138°$

This can come from using the reference angle from arcsine to get the second quadrant answer, or from graphing. Keep in mind that you are not allowed a CAS on the IB exam, and the nsolve command on the numerical TI-nspire only finds one answer at a time.

11. If A is an obtuse angle in a triangle and sin A = $\frac{5}{13}$**, calculate the exact value of sin 2A.**

The formula for $\sin 2A$ requires the value of $\cos A$, so we find that first. Note that the angle A is obtuse, so its cosine is negative.

$\sin^2 A + \cos^2 A = 1$

$\left(\frac{5}{13}\right)^2 + \cos^2 A = 1$

$\cos^2 A = 1 - \frac{25}{169} = \frac{144}{169}$

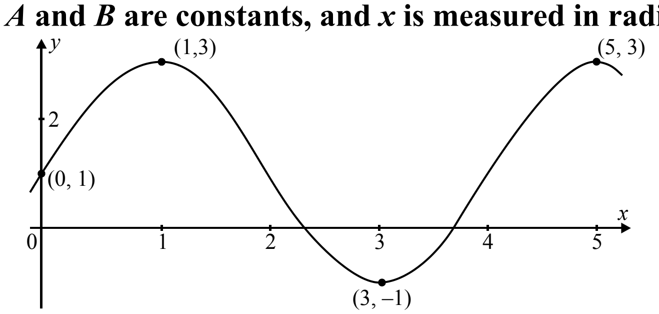
$\cos A = -\frac{12}{13}$

Then $\sin 2A = 2\sin A\cos A = 2\cdot\frac{5}{13}\cdot -\frac{12}{13} = -\frac{120}{169}$.

12. The diagram shows the graph of the function *f* given by

$$f(x) = A\sin\left(\frac{\pi}{2}x\right) + B,$$

for 0 ≤ *x* ≤ 5, where *A* and *B* are constants, and *x* is measured in radians.



The graph includes the points (1, 3) and (5, 3), which are maximum points of the graph.

(a) Write down the values of $f(1)$ **and** $f(5)$**.**

(b) Show that the period of *f* is 4.

The point (3, –1) is a minimum point of the graph.

(c) Show that $A = 2$, and find the value of B.

(d) Show that $f'(x) = \pi \cos\left(\frac{\pi}{2}x\right)$.

The line $y = k - \pi x$ is a tangent line to the graph for $0 \le x \le 5$.

(e) Find

(i) the point where this tangent meets the curve;

(ii) the value of k.

(f) Solve the equation $f(x) = 2$ for $0 \le x \le 5$.

(a) $f(1) = 3, f(5) = 3$

(b) Period = distance between two consecutive maxima.

$5 - 1 = 4 =$ period

(c) A is the amplitude, half the total height.

$$A = \frac{3-(-1)}{2} = \frac{4}{2} = 2$$

$$B = \frac{\max + \min}{2} = \frac{3+(-1)}{2} = \frac{2}{2} = 1$$

(d) $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$

$$f'(x) = 2\cos\left(\frac{\pi}{2}x\right) \cdot \frac{\pi}{2} + 0 = \pi\cos\left(\frac{\pi}{2}x\right)$$

(e) (i) The slope of the tangent line is $-\pi$.

$$f'(x) = \pi\cos\left(\frac{\pi}{2}x\right) = -\pi$$

$$\cos\left(\frac{\pi}{2}x\right) = -1$$

$$\frac{\pi}{2}x = \pi \quad \text{or} \quad \frac{\pi}{2}x = 3\pi$$

$x = 2$ or $x = 6$ (outside the domain)

So the only solution in the given domain is $x = 2$. Since $f(2) = 2\sin(\pi) + 1 = 1$, the point is (2, 1).

(ii) The tangent line passes through (2, 1), so $1 = k - 2\pi$, and $k = 2\pi + 1$.

(f) $2\sin\left(\frac{\pi}{2}x\right) + 1 = 2$

$$\sin\left(\frac{\pi}{2}x\right) = \frac{1}{2}$$

$$\frac{\pi}{2}x = \frac{\pi}{6},\quad \frac{\pi}{2}x = \frac{5\pi}{6},\quad \frac{\pi}{2}x = \frac{13\pi}{6},\quad \frac{\pi}{2}x = \frac{17\pi}{6}$$

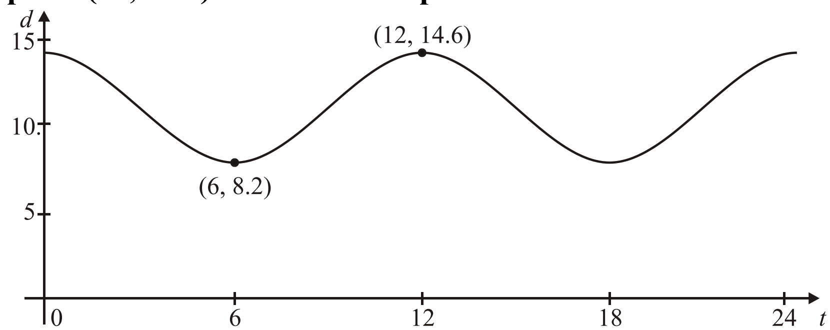
$$x = \frac{1}{3},\quad x = \frac{5}{3},\quad x = \frac{13}{3},\quad x = \frac{17}{3}$$

That last one is too large for the domain, so $x = \frac{1}{3}, \frac{5}{3}, \frac{13}{3}$

13. **A formula for the depth d metres of water in a harbour at a time t hours after midnight is**

$$d = P + Q\cos\left(\frac{\pi}{6}t\right),\ 0 \le t \le 24,$$

where P and Q are positive constants. In the following graph the point (6, 8.2) is a minimum point and the point (12, 14.6) is a maximum point.



(a) **Find the value of**

(i) Q;

(ii) P.

(b) **Find the *first* time in the 24-hour period when the depth of the water is 10 metres.**

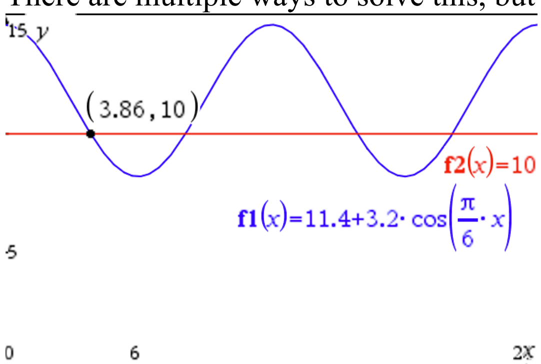
(c) **(i)** **Use the symmetry of the graph to find the *next* time when the depth of the water is 10 metres.**

(ii) **Hence find the time intervals in the 24-hour period during which the water is less than 10 metres deep.**

(a) (i) $Q = \dfrac{14.6 - 8.2}{2} = 3.2$

(ii) $P = \dfrac{14.6 + 8.2}{2} = 11.4$

(b) There are multiple ways to solve this, but the simplest one is graphing.

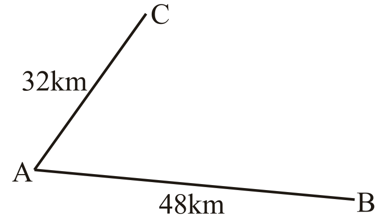


$t \approx 3.86$ hours

(c) (i) Using symmetry is a requirement here. Since the first time this depth happens is 3.86 hours after the first maximum, the second time will be 3.86 hours before the second one. The period is $\dfrac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot \dfrac{6}{\pi} = 12$ and the first maximum is at $t = 0$, so the second is at $t = 12$. Then $12 - 3.86 \approx 8.14$ is the next time the depth is 10 m.

(ii) The symmetry keeps working. The first interval is $3.86 < t < 8.14$, based on the results we already have. The second interval will start at $12 + 3.86 = 15.86$ and end at $24 - 3.86 = 20.14$. To three significant figures, that's $15.9 < t < 20.1$.

14. **Town A is 48 km from town B and 32 km from town C as shown in the diagram.**

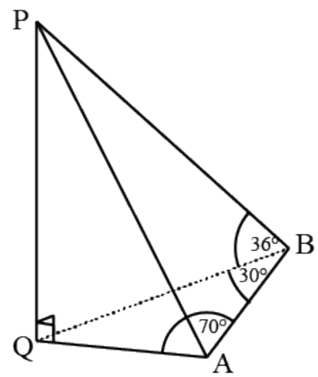


Given that town B is 56 km from town C, find the size of angle $C\hat{A}B$ to the nearest degree.

Law of cosines: $\cos A = \dfrac{32^2 + 48^2 - 56^2}{2 \cdot 32 \cdot 48}$

$mC\hat{A}B \approx 86°$, to the nearest degree

15. **The diagram shows a vertical pole PQ, which is supported by two wires fixed to the horizontal ground at A and B.**



BQ = 40 m; $P\hat{B}Q$ = 36°; $B\hat{A}Q$ = 70°; $A\hat{B}Q$ = 30°

Find

(a) the height of the pole, PQ;

(b) the distance between A and B.

(a) Using triangle BQP, $\tan 36° = \dfrac{PQ}{40}$, so $PQ = 40 \tan 36° \approx 29.1$ m.

(b) $mB\hat{Q}A = 180° - 30° - 70° = 80°$

Using triangle BQA and the law of sines, $\dfrac{AB}{\sin 80°} = \dfrac{40}{\sin 70°}$, and $AB = \dfrac{40 \sin 80°}{\sin 70°} \approx 41.9$ m.

16. **A triangle has sides of length 4, 5, 7 units. Find, to the nearest tenth of a degree, the size of the largest angle.**

The largest angle is across from the longest side.

$\cos\theta = \dfrac{4^2 + 5^2 - 7^2}{2 \cdot 4 \cdot 5}$, and $\theta \approx 102°$

17. Two boats A and B start moving from the same point P. Boat A moves in a straight line at 20 km h^{-1} and boat B moves in a straight line at 32 km h^{-1}. The angle between their paths is 70°.

Find the distance between the boats after 2.5 hours.

After 2.5 hours, boat A has traveled 50 km and B has gone 80 km. Use the law of cosines.

$AB^2 = 50^2 + 80^2 - 2 \cdot 50 \cdot 80 \cdot \cos 70^\circ$

$AB \approx 78.5$ km

18. In a triangle ABC, AB = 4 cm, AC = 3 cm and the area of the triangle is 4.5 cm^2.

Find the two possible values of the angle $\mathrm{B\hat{A}C}$.

$\text{Area} = \frac{1}{2} ab \sin C$

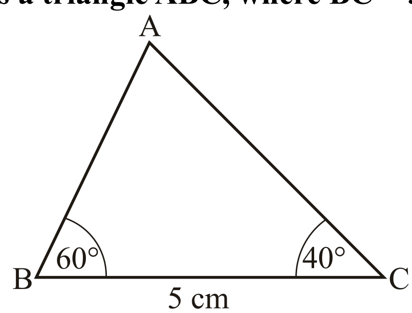
$4.5 = \frac{1}{2} \cdot 4 \cdot 3 \sin \mathrm{B\hat{A}C} = 6 \sin \mathrm{B\hat{A}C}$

$\sin \mathrm{B\hat{A}C} = \frac{4.5}{6} = 0.75$

$m\mathrm{B\hat{A}C} = 48.6^\circ$ or 131°

The second answer is the supplement of the first one; sine is positive in both quadrants I and II.

19. The following diagram shows a triangle ABC, where BC = 5 cm, $\hat{\mathrm{B}} = 60°$, $\hat{\mathrm{C}} = 40°$.



(a) Calculate AB.

(b) Find the area of the triangle.

(a) $m\mathrm{B\hat{A}C} = 180^\circ - 60^\circ - 40^\circ = 80^\circ$

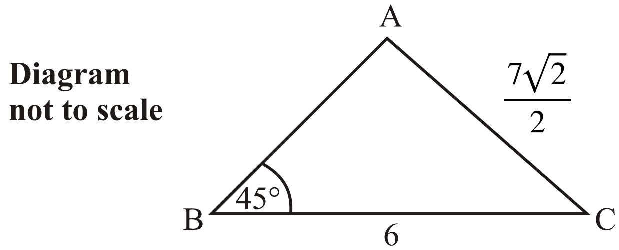
$$\frac{AB}{\sin 40^\circ} = \frac{5}{\sin 80^\circ}$$

$$AB = \frac{5 \sin 40^\circ}{\sin 80^\circ} \approx 3.26 \text{ cm}$$

(b) $\frac{1}{2} \cdot 3.26 \cdot 5 \sin 60^\circ \approx 7.07$ cm^2

This problem actually had a one mark penalty for missing units. It's a little unusual on an IB exam, but it does happen. You should give units with your answer if there are units in the question.

20. The diagram shows a triangle ABC in which AC = $\frac{7\sqrt{2}}{2}$, BC = 6, $A\hat{B}C$ = 45°.



(a) Use the fact that sin 45° = $\frac{\sqrt{2}}{2}$ to show that $\sin B\hat{A}C = \frac{6}{7}$**.**

The point D is on (AB), between A and B, such that $\sin B\hat{D}C = \frac{6}{7}$**.**

(b) (i) Write down the value of $B\hat{D}C + B\hat{A}C$**.**

(ii) Calculate the angle BCD.

(iii) Find the length of [BD].

(c) Show that $\frac{\text{Area of }\Delta BDC}{\text{Area of }\Delta BAC} = \frac{BD}{BA}$**.**

(a) Using the law of sines, $\frac{\sin B\hat{A}C}{6} = \frac{\sin 45^\circ}{\frac{7\sqrt{2}}{2}}$.

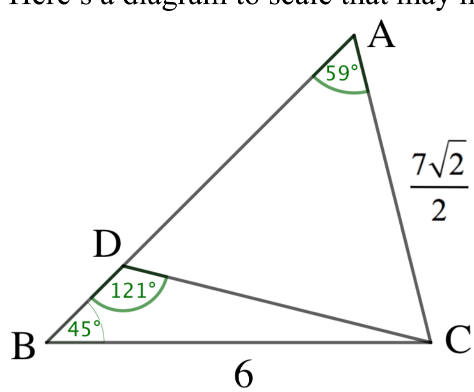
$$\frac{\sin B\hat{A}C}{6} = \frac{\frac{\sqrt{2}}{2}}{\frac{7\sqrt{2}}{2}} = \frac{1}{7}$$

$$\sin B\hat{A}C = \frac{6}{7}$$

Because this is a "show that," it must be done with exact values, not approximations.

(b) (i) Since the two angles can't be the same (because of the location of D), having the same sine means they're supplementary, and the sum is 180°.

(ii) If $\sin\theta = \frac{6}{7}$, then $\theta \approx 59.0^\circ$ or 121°. Because D is closer to B than A is, $m B\hat{D}C \approx 121^\circ$. Here's a diagram to scale that may help.



Therefore $m B\hat{C}D \approx 180^\circ - 45^\circ - 121^\circ = 14.0^\circ$

(iii) $\dfrac{BD}{\sin 14.0°} = \dfrac{6}{\sin 121°}$, so $BD = \dfrac{6\sin 14.0°}{\sin 121°} \approx 1.69$

(c) $\dfrac{\text{Area of } \Delta\text{BDC}}{\text{Area of } \Delta\text{BAC}} = \dfrac{\frac{1}{2} \cdot \text{BD} \cdot \text{DC} \cdot \sin \text{B}\hat{\text{D}}\text{C}}{\frac{1}{2} \cdot \text{BA} \cdot \text{AC} \cdot \sin \text{B}\hat{\text{A}}\text{C}}$

We already know that $\sin \text{B}\hat{\text{D}}\text{C} = \sin \text{B}\hat{\text{A}}\text{C} = \dfrac{6}{7}$, and $m\text{A}\hat{\text{D}}\text{C} \approx 180° - 121° = 59.0°$. That makes ΔADC isosceles, and $\text{DC} = \text{AC} = \dfrac{7\sqrt{2}}{2}$.

$\dfrac{\text{Area of } \Delta\text{BDC}}{\text{Area of } \Delta\text{BAC}} = \dfrac{\frac{1}{2} \cdot \text{BD} \cdot \text{DC} \cdot \sin \text{B}\hat{\text{D}}\text{C}}{\frac{1}{2} \cdot \text{BA} \cdot \text{AC} \cdot \sin \text{B}\hat{\text{A}}\text{C}} = \dfrac{\frac{1}{2}\text{BD} \cdot \frac{7\sqrt{2}}{2} \cdot \frac{6}{7}}{\frac{1}{2}\text{BA} \cdot \frac{7\sqrt{2}}{2} \cdot \frac{6}{7}} = \dfrac{\text{BD}}{\text{BA}}$. Q.E.D.

21. The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. QR = 9 km, $\text{P}\hat{\text{Q}}\text{R} = 35°$, $\text{P}\hat{\text{R}}\text{Q} = 25°$.

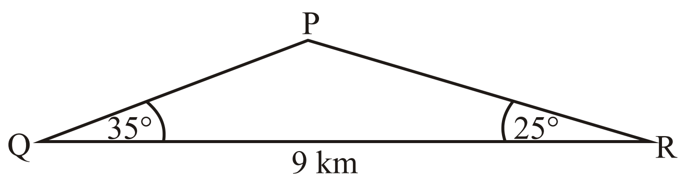
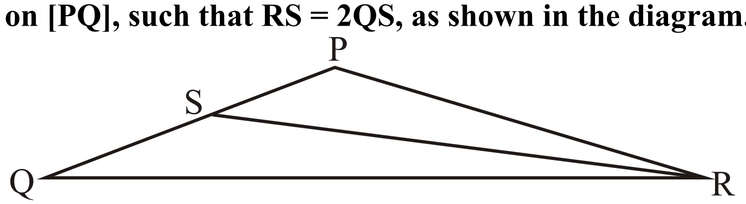


Diagram not to scale

(a) Find the length PR.

(b) Tom sets out to walk from Q to P at a steady speed of 8 km h^{-1}. At the same time, Alan sets out to jog from R to P at a steady speed of *a* km h^{-1}. They reach P at the same time. Calculate the value of *a*.

(c) The point S is on [PQ], such that RS = 2QS, as shown in the diagram.



Find the length QS.

(a) $180° - 35° - 25° = 120° = m\text{Q}\hat{\text{P}}\text{R}$

$\dfrac{PR}{\sin 35°} = \dfrac{9}{\sin 120°}$

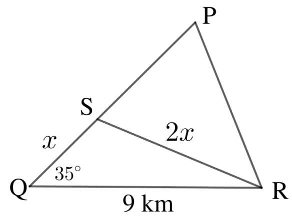
$PR = \dfrac{9\sin 35°}{\sin 120°} \approx 5.96$ km

(b) Since they arrive at the same time, $\dfrac{\text{PQ}}{8} = \dfrac{\text{PR}}{a}$.

$\dfrac{PQ}{\sin 25°} = \dfrac{9}{\sin 120°}$, so $PQ = \dfrac{9\sin 25°}{\sin 120°} \approx 4.39$ km

Then $\dfrac{4.39}{8} = \dfrac{5.96}{a}$, and $a = \dfrac{8 \cdot 5.96}{4.39} \approx 10.9$ km h^{-1}.

(c) If RS = 2QS, it may help to label those lengths appropriately.



Since there are labels on three sides and an angle, the law of cosines may help.

$(2x)^2 = x^2 + 9^2 - 2 \cdot x \cdot 9 \cdot \cos 35°$

$4x^2 = x^2 + 81 - 18x \cos 35°$

That’s actually a quadratic in x. I like nsolve for this.

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nSolve(4·x^2=x^2+81−18·x·cos(35°),x)
                                   3.29051
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QS ≈ 3.29 km