Some calculus concepts, explained

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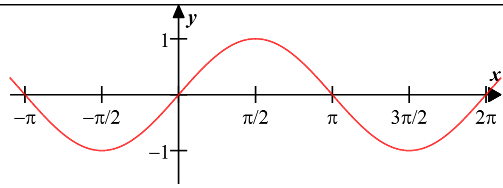
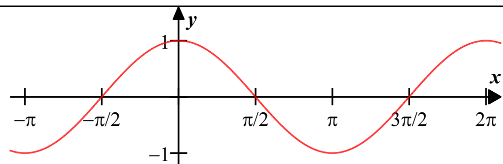
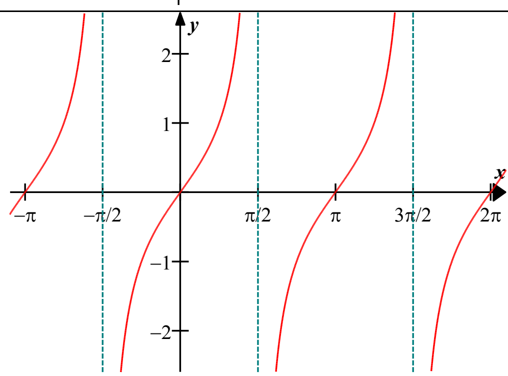
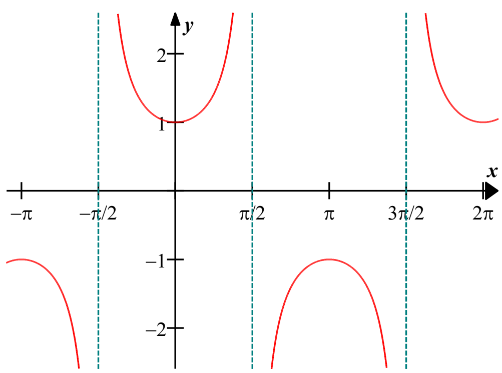
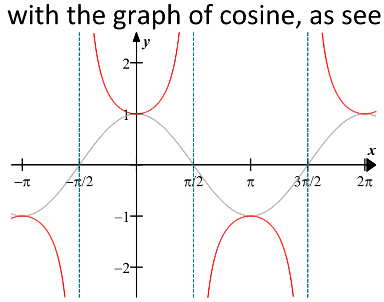
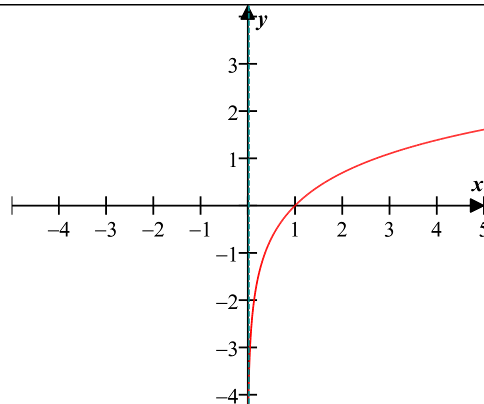
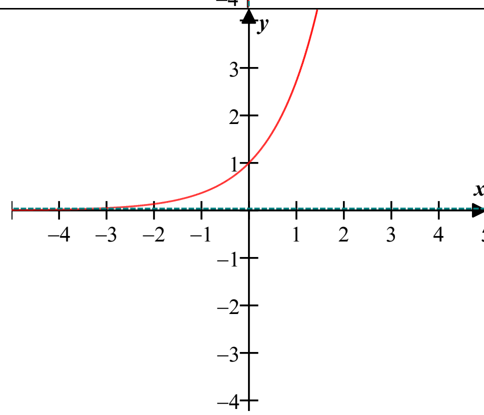
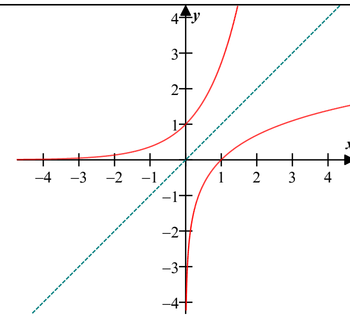
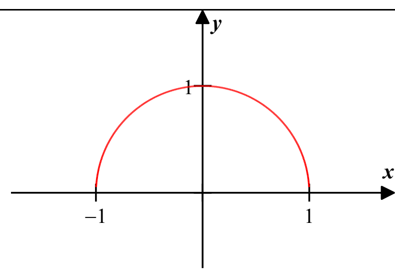
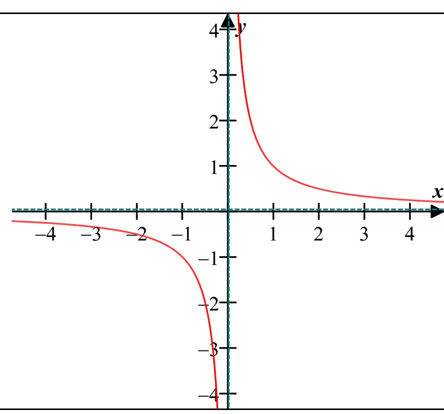
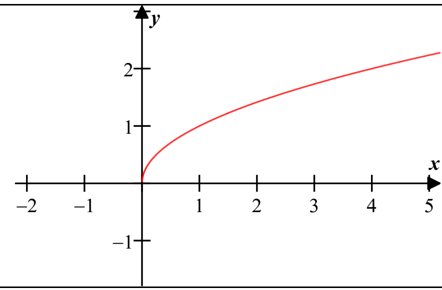
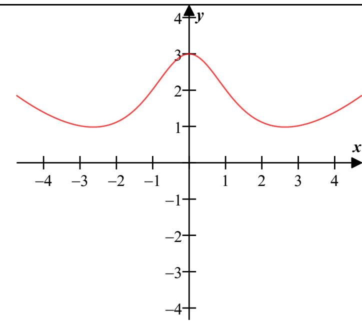
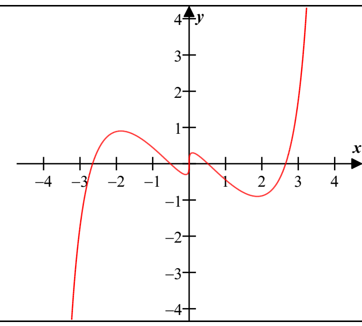
Graphs of basic functions

Fact: Knowing what some basic functions look like can really help with some problems. The simplest ones, like $y = x^2$, aren't given here.

Function	Graph	Important Facts
$y = \sin x$		Yes, I'm sure you know what sine looks like. Note that this can help you see that $\sin 0 = 0$, and so on; sine "starts" at the origin, in the middle of its range, and goes up. Its period is 2π, its domain is $\mathbb{R}$, and its range is $[-1, 1]$. Important points happen every ¼ of the period, or $\frac{\pi}{2}$.
$y = \cos x$		Cosine, on the other hand, starts at the top of its range when $x = 0$. The period, domain, and range are the same as for sine.
$y = \tan x$		Tangent passes through the origin, since $\tan 0 = \ 0$. The asymptotes are at odd multiples of $\frac{\pi}{2}$. Notice that the slope of tangent at its x-intercepts is not particularly flat. If you check the value of its derivative, you'll discover that the slope there is 1. The graph of cotangent is similar, but slopes down rather than up, and has asymptotes at integer multiples of π.
$y = \sec x$		Secant is the reciprocal of cosine, so its asymptotes occur where cosine is zero, at odd multiples of $\frac{\pi}{2}$ — just like tangent! That's because tangent is sine divided by cosine; it *also* has cosine in the denominator. The other cool thing about secant is how it "fits" together with the graph of cosine, as seen below.  Cosecant looks much the same, except that the asymptotes go where sine is zero.

$y = \ln x$		The natural logarithm is the log base *e*, but all logarithm graphs look pretty much the same. Since $\ln 1 = 0$ (see the section on values to know for more information on that), the *x*-intercept is 1. Since $\ln 0$ is undefined, there is no *y*-intercept. The *y*-axis is a vertical asymptote of the function. It's handy to know that logarithms grow more slowly than *any* polynomial. Eventually even $y = 0.00001x$ will be above this graph.
$y = e^x$		 As you can see in the graph above, the natural exponential function and the natural logarithm function are inverses, and so are reflections of each other in $y = x$. The x-axis is a horizontal asymptote, and the y-intercept is 1. Also, exponential functions grow faster than any polynomial; eventually this function would end up above $y = x^{100}$.
$y = \sqrt{1 - x^2}$		Here we've got a semicircle. This time I used 1 for the radius to get half of the unit circle. In general, $y = \sqrt{r^2 - x^2}$ would be a semicircle centered at the origin with radius r. There was once a slope field problem on the no-calculator multiple choice where knowing this shape helped.
$y = \dfrac{1}{x}$		This has both of the axes as asymptotes. Notice that it's always sloping down. Its derivative is negative.

$y = \sqrt{x}$		The domain of this one is $x \geq 0$; yes, zero *is* in the domain.
Even functions	Even functions can be defined both geometrically and algebraically. Geometrically, a function is even if it is symmetric about the y-axis. Algebraically, this translates to $f(-x) = f(x)$. The most common examples are $y = x^2$ (see the "evenness"?) and $y = \cos x$. Since you already know those two, I've included a *different* example here.	
Odd functions	Odd functions, on the other hand, are symmetric about the *origin*. Algebraically, $f(-x) = -f(x)$. I remember it as "*O*, odd, origin." The most common examples are $y = x$, $y = x^3$, and $y = \sin x$. I'm putting a fourth one to the right. Yeah, you'll never figure out what I graphed there.[1]	

Fact: Inverse functions are defined like this: f and g are inverse functions if $f\big(g(x)\big) = g\big(f(x)\big) = x$ for all x in the appropriate domains. We then write that $g(x) = f^{-1}(x)$. Their graphs, as seen in the entry for exponential functions above, are reflections of each other in the line $y = x$.

Def'n: The line $y = a$ is a ***horizontal asymptote*** of the graph of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = a$ or $\lim_{x\to-\infty} f(x) = a$. The line $x = a$ is a ***vertical asymptote*** of the graph of $y = f(x)$ if either $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Note that the limit can be one-sided rather than two-sided.

[1] It's three perfectly ordinary functions combined in some way, if you're really bored and have way too much time on your hands. Actually, on second thought, you should spend that time *studying for this exam!* STOP trying to figure it out!

Some important precalculus facts to have memorized

Formulas from geometry (useful with related rates and volumes of solids)

Area of a triangle: $A = \frac{1}{2}bh = \frac{1}{2}ab\sin C$	***Area of a trapezoid***: $A = \frac{1}{2}h(b_1 + b_2)$
Area of a circle: $A = \pi r^2$	***Circumference of a circle***: $C = 2\pi r$
Volume of a cylinder: $V = \pi r^2 h$	***Volume of a cone***: $V = \frac{1}{3}\pi r^2 h$
Volume of a sphere: $V = \frac{4}{3}\pi r^3$	***Surface area of a sphere***: $SA = 4\pi r^2$
The Pythagorean Theorem: In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs; i.e., $\mathrm{leg}^2 + \mathrm{leg}^2 = \mathrm{hyp}^2$.	

Sine, cosine, and tangent values, for angles in radians and degrees

θ	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	$0°$	0	1	0
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$45°$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	$90°$	1	0	undefined
π	$180°$	0	-1	0
$\frac{3\pi}{2}$	$270°$	-1	0	undefined
2π	$360°$	0	1	0

Fact: Tangent is sine divided by cosine; if you know the first two columns above, you can always calculate the corresponding value of tangent in very little time.

Fact: The values of sine are positive when y is positive, in quadrants I and II, and negative elsewhere; cosine values are positive where *x* is positive, in quadrants I and IV. That's "ASTC" in quadrant order.

Fact: ***Exponential functions*** and ***logarithmic functions*** are inverses of each other.

Fact: These values come up *a lot*, and you should know them automatically.

$$\ln 1 = 0 \qquad \ln e = 1 \qquad \ln e^2 = 2 \qquad e^0 = 1 \qquad e^1 = e \qquad e^{-1} = \frac{1}{e}$$

Laws of Logarithms

$$\log_b M + \log_b N = \log_b MN \qquad \log_b M - \log_b N = \log_b \left(\frac{M}{N}\right) \qquad \log_b(M^p) = p \log_b M$$

Be careful: There is no general rule like this for separating the argument of a logarithm that has terms added or subtracted. For instance, $\ln(1 + x^2)$ does *not* break down into two separate logarithms.

Defining the derivative of a function $f(x)$

Def'n: The ***derivative*** of a function $f(x)$ is a formula for calculating its slope at any point on the function.

Def'n: The ***definition of the derivative*** of a function $f(x)$ is $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$. The derivative at $x = a$ can be written two different ways: $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$.

Fact: The fraction whose limit is taken in that formula is known as the ***difference quotient***, and it represents the slope between two points on the curve very close together: $(x, f(x))$ and $(x + h, f(x + h))$.

Computing the derivative of a function $f(x)$

Fact: We hardly ever use the definition of derivative to actually find derivatives of functions. Instead, we use the derivative rules below.

Rule	***Formula***	***Comments***
Power	$\frac{d}{dx}[x^n] = nx^{n-1}$	This *only* works when the base is a variable and the power is a number.
Constant Multiple	$\frac{d}{dx}[kf(x)] = kf'(x)$	If you have a constant times a function, the constant just comes along for the ride. This makes sense, because multiplying by a constant gives a vertical dilation; all of the slopes are k times as great as before.
Sum and Difference	$\frac{d}{dx}[f(x) \pm g(x)]$ $= f(x) \pm g'(x)$	This one lets you do what you would certainly do already — put a plus sign between the separate derivatives
Product	$\frac{d}{dx}[f(x)g(x)]$ $= f(x)g'(x) + g(x)f'(x)$	Here's where it starts to get more challenging. I remember this as first times the derivative of the second plus second times the derivative of the first, or $\text{1st} \cdot d\ \text{2nd} + \text{2nd} \cdot d\ \text{1st}$.
Quotient	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ $= \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$	This is the old Lo-d-Hi.[2] Remember that "low" is on the outside of both top and bottom. It's $\frac{\text{lo} \cdot d\ \text{hi} - \text{hi} \cdot d\ \text{lo}}{\text{lo} \cdot \text{lo}}$.

[2] Listen to the Blood, Sweat & Tears song.

Rule	Formula	Comments
Chain	$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'(g(x)) \cdot g'(x)$	Aretha![3] The chain rule is vital, as you will recall if you've ever done an entire assignment without remembering to use it at all. Whenever there's an "inside" function — whenever the function doesn't *just* have *x* in it — you take the derivative of the outside function and then multiply by the derivative of the inside function. Every function has its derivative show up once.
Sine	$\frac{d}{dx}[\sin x] = \cos x$	The trigonometric derivatives need to be memorized completely. Don't waste time trying to figure out the derivative of tangent by rewriting as a fraction of sine and cosine; you don't have that much time to spare. Recall that all of the derivatives of the "co" functions are negative. This is true with the inverse trig functions as well.
Cosine	$\frac{d}{dx}[\cos x] = -\sin x$	
Tangent	$\frac{d}{dx}[\tan x] = \sec^2 x$	
Cotangent	$\frac{d}{dx}[\cot x] = -\csc^2 x$	
Secant	$\frac{d}{dx}[\sec x] = \sec x \tan x$	
Cosecant	$\frac{d}{dx}[\csc x] = -\csc x \cot x$	
Natural Exponential	$\frac{d}{dx}[e^x] = e^x$	This is the easiest derivative of all. Don't make any mistakes with it. If the function needs the chain rule, it looks like $\frac{d}{dx}\left[e^{\text{stuff}}\right] = e^{\text{stuff}} \cdot \frac{d}{dx}[\text{stuff}]$.
General Exponential	$\frac{d}{dx}[a^x] = a^x \ln a$	This one is very similar to the derivative of e^x, but because the base isn't e, you multiply by the natural logarithm of the base. (If the base *were* e, then $\ln e$ would be 1 and disappear.)
Natural Logarithm	$\frac{d}{dx}[\ln x] = \frac{1}{x}$	Make sure you don't get it backwards. The derivative of natural log is $\frac{1}{x}$, not the other way around. To take the *derivative* of $\frac{1}{x}$, you rewrite it as x^{-1} and use the power rule.
General Logarithm	$\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$	Similarly to the exponential function with base a, there's an extra $\ln a$. This time it's in the denominator, with the x.

[3] Listen to the Aretha Franklin song. Or maybe *f(u)* is more your style.

Rule	***Formula***	***Comments***
Inverse sine ($\sin^{-1}(\,)$)	$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$	The most likely of these to appear are arcsine and arctangent, distantly followed by arcsecant. The others, like ordinary trig functions, have negative derivatives when they start with "co."
Inverse tangent ($\tan^{-1}(\,)$)	$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$	
Inverse secant ($\sec^{-1}(\,)$)	$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	
Inverse cosine ($\cos^{-1}(\,)$)	$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$	
Inverse cotangent ($\cot^{-1}(\,)$)	$\frac{d}{dx}[\operatorname{arccot} x] = \frac{-1}{1+x^2}$	
Inverse cosecant ($\csc^{-1}(\,)$)	$\frac{d}{dx}[\operatorname{arccsc} x] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	
Implicit differentiation	This doesn't really have a formula. For a discussion of implicit differentiation, see the section on related rates problems.	
Derivative of an inverse function	$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$	Is it *really* important that you have this formula memorized? No. The key thing here is that the slopes of inverse functions are reciprocals of each other (since $\frac{\Delta y}{\Delta x}$ becomes $\frac{\Delta x}{\Delta y}$ when the variables are interchanged), but at the points which correspond to each other, NOT at the points with the same x-value. In other words, the slope at (a, b) is the reciprocal of the slope at (b, a) on the inverse function.

Finding slope of a function $f(x)$ at a point
Finding instantaneous rate of change of a function $f(x)$
Finding a tangent line to a curve $y = f(x)$
Finding a normal line to a curve $y = f(x)$

Fact: The ***slope of a curve at a point*** is the same thing as the ***slope of the tangent line*** at that point and as the ***instantaneous rate of change*** of the function there. This value is found using the ***derivative***.

Fact: The ***average rate of change*** of a function over an interval is just the slope between the endpoints.

Fact: To find an equation for a line, you generally want two facts: a point and the slope. If you need the equation for a tangent line, the difficulty is that you only get one point on the curve, and to find a slope, it generally takes two points. That's what the derivative is for. Not to belabor what I wrote earlier, the derivative is a formula for the slope at a point. So take the derivative and use the given x-value (or the one you found, whichever) to get a *numerical* slope. I like to start with point-slope form: $y - y_1 = m(x - x_1)$.

Be careful: Many people are occasionally tripped up by their propensity to write the *formula* for the derivative where a *numerical value of slope* should go. If you're asked for a line, it had better look like an equation for a line when you're done.

Def'n: A ***tangent line*** is a line that passes through a point on the curve and has the same slope as the curve does at that point. (In other words, its slope is the value of the derivative at that point.)

Def'n: A ***normal line*** is a line perpendicular to the tangent at the point of tangency. If you've already found the slope of the tangent, the slope of the normal is the negative reciprocal of that value, and the point used would be the same.

Fact: A tangent line can be used to approximate values of a function. If you know the tangent line to a curve at, say, $x = a$, then you can approximate values of the function for x near a using the line. This is useful if it's much more difficult to substitute a number into the actual function than it is to use a linear one. And it's pretty much always easier to use a line.

Finding limits[4] using L'Hôpital's rule

Fact: Sometimes when you attempt to evaluate a limit by substituting a value, you get a "nonanswer." These are called ***indeterminate forms***. Getting one does *not* necessarily mean that the limit does not exist. In fact, you have yet to *determine it*. (Indeterminate = still have to determine it, get it?)

Fact: There are several different sorts of indeterminate forms, and not all of them work with L'Hôpital's rule. For now, you don't need to worry about the ones that don't. If it matters a lot to you, ask about it after the AP Calculus exam, and I'll tell you about them again.

Rule: Suppose that when you attempt to evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ by substituting in the value of a for x, you get one of these forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$, $\frac{\infty}{-\infty}$, or $\frac{-\infty}{-\infty}$. These are all indeterminate, and the limit may or may not exist. Under those circumstances, ***L'Hôpital's Rule*** says that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the second limit exists. On the AP Calculus exam, you *must* show the limits of both numerator and denominator before using it.

Fact: This is *not* the same as the quotient rule; you take the derivatives of numerator and denominator separately.

Fact: If, when you try substituting a into the new fraction, you still get one of those five fractional indeterminate forms, you can use L'Hôpital's again. And again. And again. As long as you see that you're making progress, anyway. If you aren't getting closer to an answer, try something else. If at any point in this process you get a fraction in which the either the numerator or denominator is a nonzero constant, *don't* use the rule again! It no longer applies — and you no longer need it to answer the question.

[4] Listen to the Etta James song. (I *love* this version of the song, and I couldn't find any other vaguely logical place to attach a link.)

Finding extreme values of a function $f(x)$
Finding where a function $f(x)$ is increasing or decreasing

Def'n: ***Extreme values*** (a/k/a extrema) are maximum and minimum values.

Fact: ***Extrema*** can *only* occur at critical numbers and at endpoints.

Def'n: ***Critical numbers*** are values of x in the domain of a function at which its derivative is either zero or undefined.

Def'n: A function f is ***increasing*** on an interval if, for $x_1 < x_2$, $f(x_1) < f(x_2)$. In other words, as x gets bigger, y gets bigger. ***Decreasing*** is defined similarly, but as x gets bigger, y gets smaller.

Fact: If $f'(x) > 0$ on an interval, then $f(x)$ is ***increasing*** there; if $f'(x) < 0$ on an interval, then $f(x)$ is ***decreasing*** there. Give the largest intervals possible; you can include the endpoints if the function is continuous.

The First Derivative Test: If f is a continuous function and c is a critical number of f, then
(a) if f' changes signs from positive to negative at $x = c$, then $f(c)$ is a ***local maximum*** of f;
(b) if f' changes signs from negative to positive at $x = c$, then $f(c)$ is a ***local minimum*** of f; and
(c) if f' does not change signs at $x = c$, then $f(c)$ is not an extreme value of f.
Also, if $f' > 0$ leading up to the right endpoint of an interval, then that endpoint is a local maximum.
If $f' > 0$ leading away from the left endpoint of an interval, then that endpoint is a local minimum. In each case, the opposite is true for $f' < 0$.

Def'n: An ***absolute maximum*** (***global maximum***) of f is a y-value $f(c)$ which is the largest y-value for any x in the domain of the function. It may occur at more than one x-value. Similarly, an ***absolute minimum*** (***global minimum***) of f is a y-value$f(c)$ which is the smallest y-value.

Def'n: A ***local maximum***, also called a ***relative maximum***, is the largest *y*-value in some interval, but not necessarily in the entire domain; a ***local minimum*** (***relative minimum***) is the smallest.

<u>Process:</u>
1. Find the derivative, $f'(x)$.
2. Find values of *x* at which the derivative is 0 or undefined (critical numbers).
3. Set up a number line, labeled f' (use the actual name of the function, which might not be f). If the interval has endpoints, the number line must only go that far. Mark the critical numbers on the f' number line.
4. Check a value of x in each interval in f' to determine if f' is positive or negative there.
5. Where $f' > 0$, f is increasing, and where $f' < 0$, f is decreasing.
6. Where f' changes signs (at a place where the function is continuous), f has either a maximum or minimum. Endpoints can also be extrema. See the first derivative test above.
7. To determine the absolute extrema, check the actual y-values at the relative extrema and at the endpoints to find the largest and smallest values.

The Second Derivative Test: Suppose that c is a critical number of f.

- If $f'(c) = 0$ and $f''(x) > 0$, then *f* has a ***local minimum*** at $x = c$.
- If $f'(c) = 0$ and $f''(x) < 0$, then *f* has a ***local maximum*** at $x = c$.

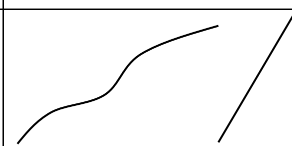
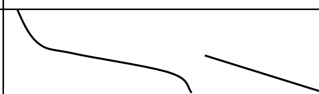
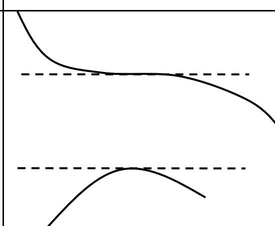
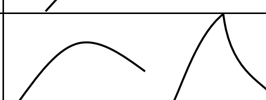
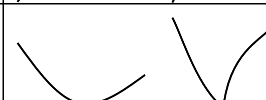
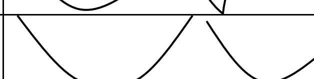
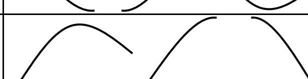
This works because of concavity. You start with the fact that f' is zero, which means there's a horizontal tangent. If f'' is positive there, then the graph of f is concave up (like a cup), and you must be at the bottom of the cup. If f'' is negative where f' is zero, then the downward concavity (like a frown) puts the point at the top of the curve.

Fact: The second derivative test is mainly used when you can't use the first derivative for some reason, either because you have no way to find the value of the first derivative on either side of the critical number, or because the derivative depends on the values of both x and y, and changing the value of x also changes the value of y that would go into the derivative.

Be careful to answer the question asked. It may ask for x-coordinates, values of f (that is, y-values), or points.

Relating f' and f'' to the graph of f

Fact: If you are given a graph of f, f', or f'' and have to draw or find the graph of one of the others, look first for places where something is zero. For instance, at a maximum of f, you generally have a zero slope, leading to a zero *value* for f'. If f'' is zero at a particular value of x, the graph of f' will have a horizontal tangent, and possibly a maximum or minimum. After you've accounted for those, look at where increasing and decreasing behaviors of a function give positive and negative values of its derivative, respectively.

f	f'	f''	**Example graphs**
The original function	The (first) derivative of f	The second derivative of f	
	Shows the slope of f	Shows the concavity of f	
		Shows slope of f', since it's the derivative of f'	
increasing (on an interval)	positive (above the *x*-axis)	no information Notice that the curve's concavity might change, but *y* is increasing the whole time.	
decreasing (on an interval)	negative (below the *x*-axis)	no information This is the same situation as above, only *y* is decreasing.	
horizontal tangent (at a point)	zero	no information	
relative maximum (at a point)	zero or undefined; changes signs + to –	negative (below the *x*-axis) or undefined	
relative minimum (at a point)	zero or undefined; changes signs – to +	positive (above the *x*-axis) or undefined	
concave up (on an interval)	increasing	positive (above the *x*-axis)	
concave down (on an interval)	decreasing	negative (below the *x*-axis)	
possible point of inflection (at a point)	no information	zero, or possibly undefined	
point of inflection (at a point)	has relative maximum or minimum	changes signs	In the graph above, the leftmost dot is *not* at a point of inflection, although the curve is so flat there that it has no concavity. The others are.

Def'n: A **point of inflection** is a point on the graph of a function at which its concavity changes. Since the second derivative gives information about the concavity, you generally look for a point of inflection by finding zeros of the second derivative. ***Be careful***, though. It is not enough that the second derivative be zero. For there to be a point of inflection, the second derivative must *change signs*. This is equivalent to the first derivative's having a maximum or minimum.

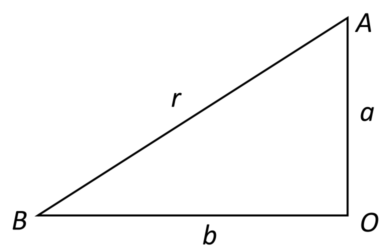
Fact: Knowing the concavity of f won't tell you if f is increasing or decreasing, and vice versa.

Solving related rates problems

Fact: Believe it or not, all related rates problems are fundamentally the same. You get an appropriate equation, you differentiate everything with respect to *t*, you plug in what you know, and you solve for what you would like to find. I'll work out an example below following this process.

Fact: You can't do these sorts of problems without ***implicit differentiation***, which is fundamental to the process. To differentiate implicitly just means that you're not solving for a particular variable first; in fact, there may be more than two variables in the equation. All of the variables are thought of as functions of *t* in the case of related rates, which means that if you differentiate r^2, for instance, the result is $2r\frac{dr}{dt}$.

Process:

0. Find a problem to do.	Two hikers begin at the same location and travel in perpendicular directions. Hiker *A* travels due north at a rate of 5 miles per hour; Hiker *B* travels due west at a rate of 8 miles per hour. At what rate is the distance between the hikers changing 3 hours into the hike?[5]
1. Read the question and see what variables are involved and what you're looking for. Translate that into symbols. A diagram may help.	North and west suggest a picture.  I'm using r for the distance, because the only reason I'd care about how far they were apart is whether their two-way radios would still work. So r is the distance the radios are from each other. I used *O* for the origin. I know that $\frac{da}{dt} = 5$ mi/h, and $\frac{db}{dt} = 8$ mi/h. I'm trying to find $\frac{dr}{dy}$ when $t = 3$ h.
2. Write an equation that relates the variables.	In this case, that's pretty clearly the Pythagorean theorem. Geometric formulas are common here, and there's a section in this document devoted to some you might want to know. Similar triangles might also come up. And sometimes the formula is just given to you. Here, it's $a^2 + b^2 = r^2$.
3. Differentiate both sides of the equation with respect to *t*, remembering to use implicit differentiation. Also, be careful	$a^2 + b^2 = r^2$ $2a\frac{da}{dt} + 2b\frac{db}{dt} = 2r\frac{dr}{dt}$

[5] From *The Humongous Book of Calculus Problems*, W. Michael Kelley, Alpha Books, 2006.

that if there are constants (for instance, if the hypotenuse were unchanging, like a ladder), that you make those derivatives zero.	In this case, I'll divide everything by 2 so that it's not so… busy. $a\dfrac{da}{dt}+b\dfrac{db}{dt}=r\dfrac{dr}{dt}$
4. Substitute in the values you know. See if you need any additional ones before you solve for the desired result.	After three hours (remember, $t=3$?), *A* will have traveled $5\ \cdot\ 3=15$ miles and *B* will have gone $8\ \cdot\ 3=24$ miles.[6] So $a\frac{da}{dt}+b\frac{db}{dt}=r\frac{dr}{dt}$ becomes $15\ \cdot 5+24\ \cdot 8=r\frac{dr}{dt}$. But I still need r.
5. If necessary, use facts from the problem to find any missing values.	In this case, I have the Pythagorean relationship to exploit. $a^2+b^2=r^2$ $15^2+24^2=r^2$ $r^2=801$ $r=\sqrt{801}$ mi
6. Finish substituting and solve for the rate you're looking for.	$15\ \cdot 5+24\ \cdot 8=\sqrt{801}\dfrac{dr}{dt}$ $267=\ \sqrt{801}\dfrac{dr}{dt}$ $\dfrac{dr}{dt}=\dfrac{267}{\sqrt{801}}\approx 9.434\text{ mi/h}$

Some important theorems

The Intermediate Value Theorem

If f is continuous on $[a,b]$, and k is any value between $f(a)$ and $f(b)$, then there is some value c between a and b such that $f(c)=k$. In English, this means that a continuous function hits every y-value between the values at the endpoints.

The Extreme Value Theorem

If a function f is continuous on the closed interval $[a,b]$, then f will have both an absolute maximum and an absolute minimum on the interval.

Rolle's Theorem

If a function f is continuous on the closed interval $[a,b]$, differentiable on the open interval (a,b), and $f(a)=f(b)$, then there is a value of c somewhere between a and b such that $f'(c)$. In other words, if a function starts and ends at the same height and it's "nice" enough, then it must have a horizontal tangent somewhere in between.

[6] I'm not sure what Mr. Kelley was thinking. *Hiking* suggests rugged terrain. Who could keep up 8 mi/h for three hours under those conditions? The Roman army used to do about 4 mi/h on their roads. Those guys were serious marchers. Twice as fast seems a little nuts to me. (Love his math explanations, though.)

The Mean Value Theorem for Derivatives

If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a c somewhere between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. In words, this says that the slope between the endpoints, also known as the slope of the secant line, must be the same as the slope of the curve at some point within the interval. It's what would let the state trooper give you a ticket for speeding even if his radar never clocked you going over the limit: if your *average* speed on the stretch of road was over the limit, then you must have been going that fast at some time. If you need help remembering the formula, it's just the slope equals the slope.

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a value c in (a, b) where $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. It's also called a "mean value theorem" because it really says the same thing as the previous one. Suppose that $F(x)$ is an antiderivative of $f(x)$. By the ***Fundamental Theorem of Calculus***, the value of the integral is $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$. Dividing both sides by $(b - a)$ gives that $\frac{F(b)-F(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x)dx = f(c)$, from above. See, it's two theorems in one! And that leads us to the most important calculus theorem of all, which is *also* two theorems in one.

The Fundamental Theorem of Calculus

As stated above, there are two parts to this one, and they're customarily labeled part 1 and part 2. However, there is not universal agreement as to *which* part is first and which is second. Here they are presented in the same order as Rogawski does.

FTC, part 1

If f is continuous on $[a, b]$ and F is any antiderivative of f on that interval, then the definite integral of f can be evaluated as $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$. This is fundamental because it shows that differentiation and integration are inverse operations; to evaluate a definite integral, first find the antiderivative. Remember that when you evaluate a definite integral this way, the upper limit of integration is substituted in first. (If you get to the top, you get to be first in line.)

FTC, part 2

If f is continuous on $[a, b]$, then the function defined by the definite integral $F(x) = \int_a^x f(t)dt$ is differentiable on the interval $[a, b]$, and its derivative is $F'(x) = \frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$. This is just as fundamental as the first part, but states the relationship between differentiation and integration in a different way: take the derivative of an integral, and you wind up with the original function f. This can be used to find the derivatives of functions defined as definite integrals. When you do that, beware of unusual limits of integration and the chain rule.

Solving kinematics problems (position, velocity, acceleration…)

Fact: ***Position*** is usually given in this sort of problem as a location on either the x- or y-axis, as in "the position of a particle traveling along the x-axis at time t is given by the function $x(t) =$" something or other. Associated with this is the idea of ***displacement***, which literally means how far "out of place" the object is — its position relative to where it started.

Def'n: ***Velocity*** is the rate of change of position. As you know, rate of change is just slope. Depending on whether you look at the slope between two points or at a single point, you can have ***average velocity*** or ***instantaneous velocity***.

Def'n: ***Average velocity*** is displacement divided by elapsed time, or change in position over change in time. This takes two points to figure out.[7]

Def'n: ***Instantaneous velocity*** is the derivative of position with respect to time: $v(t) = s'(t)$.[8] When the term velocity is used alone, it is assumed to be the instantaneous sort.

Def'n: ***Acceleration*** is the rate of change of velocity. Of course, you could have average acceleration, just as with velocity, but it's calculus class, and we are generally talking about the instantaneous change, so $a(t) = v'(t)$.

Def'n: For completeness' sake, I'll add that the derivative of acceleration is called ***jerk***, but I guarantee that you won't find that term on the AP calculus exam.

Fact: To get from position to velocity to acceleration, differentiate. The units can help you with this. Position is in distance units, like meters. Velocity would then be in something like meters per second, and acceleration in meters per second squared. To get from acceleration to velocity to position, integrate.

Fact: ***Displacement*** can be thought of as net ***distance***. Displacement can be found with the definite integral of velocity. However, if the object changes direction, just integrating will not find distance. First determine any times at which the velocity is zero. Those will divide the interval into pieces. Find the displacement on each piece. If any of those are negative (that's moving backward), make them positive before adding together to get a total distance. If you have a calculator, it's easier — just integrate the absolute value of the velocity function, also known as ***speed***.

Finding the average value of a function

Fact: The formula for the **average value of a function** is the same as the formula associated with the Mean Value Theorem for Integrals. Mean value, average value… get it?

Rule: The average value of a function is the average *height* of the function. The ***formula for the average value of f*** on the interval $[a, b]$ is $f_{avg} = \bar{f} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{\int_a^b f(x)dx}{b-a} = \frac{\text{integral}}{\text{interval}}$. Since the area of a rectangle is base × height, this works out to the area divided by the base to get the height. Remember that definite integrals come from adding up the areas of rectangles to begin with.

[7] Actually, if you have a velocity function, you could also find average velocity by using the formula for the average value of a function. If you have a position function, do the change in position divided by change in time. You'll get the same answer either way.

[8] The letter *s* is used for position because it's from the Latin *situs*, translated as position or site. Michael Spivak is the man.

Solving differential equations

Def'n: A ***differential equation*** is an equation containing one or more derivatives. A ***separable differential equation*** is one in which *x* and *y* can be put on different sides.

Fact: To ***solve a differential equation*** means to take an antiderivative (or perhaps more than one) until you find the original function.

Def'n: A ***general solution*** of a differential equation is one in which the "$+\ C$" from the antiderivative remains in the function in some form. It gives a formula for a lot of different functions which make the differential equation true.

Def'n: A ***particular solution*** of a differential equation is one in which you use information about values of the original function, first derivative, and so on, to find the value of C and therefore the one function that satisfies both the differential equation and the given ***initial condition(s)***.

Process:

1. If necessary, ***separate the variables*** by multiplying or dividing so that all of the instances of one variable are on the left side of the equation and those of the second variable are on the right. It is particularly important in this process that dx and dy (or whatever variables you have) do not end up in the denominator of a fraction. You *cannot* integrate a function with dx or dy in the denominator.
2. Write integral signs on both sides of the equation so that each side looks like an integral with correct notation. *Do not skip this step*; it shows your work and makes clear to the reader that you understand what you're doing. (Also, it is a good place to whistle.)
3. Integrate each side, using whatever techniques are necessary to find the antiderivatives.
4. As soon as the last integral sign is gone from the equation, the ***constant of integration*** ("$+\ C$") must appear, usually on the side with the dependent variable. While you could add a constant to each side, they would have to be different, and then you could subtract one to the second side and be back to a single constant. *Do not wait to do this until later*; the result of that is often an incorrect answer. You've been warned.

Sometimes you can stop after step 4. Sometimes you are given an initial condition that lets you determine the value of C. Sometimes you are asked to find "the function $y = f(x)$," which requires that you solve for y in the answer.

Fact: A very common differential equation is $y' = ky$, in which the rate of change is directly proportional to the value of the function. Its solution is $y = Ce^{kt}$. In the process of solving for y, removing a natural logarithm puts the C in a different place. This is one of the cases mentioned above in which adding C at the end will give an incorrect answer.

Finding the area between curves

Fact: The ***area between curves*** is found by integrating the top curve minus the bottom curve. That sounds ridiculously simple, and mainly it is, but there are a couple of things to be careful of.

Process:

1. Determine the limits of integration. Sometimes these are given to you as an interval on which you're to find the area. Sometimes they come from specified vertical lines, like the y-axis or $x = 3$. And sometimes you have to determine one or both limits yourself by finding the intersection(s) of the two functions involved.

   Be careful that you actually write down the equation that you're solving when you find those limits. It's usually something like $f(x) = g(x)$. The values cannot seem to come out of nowhere. However, if you have a calculator, it is perfectly permissible and sometimes absolutely necessary to solve that equation using technology.
2. Decide which function is on top. If there's a graph, this is simple. If you have to generate your own graph, though, be really careful that you don't miss any intersections. If the functions intersect more than two times, you'll have to deal with step 3.
3. If one function is not always on top, it will take more than one integral to find the area. You need one $\int_a^b (\text{top} - \text{bottom})dx$ for each separate region. Write them down! Don't leave an integral "assumed." You have to show that you know the calculus, and part of that is writing down correct calculus.
4. Integrate, either by hand if you don't have a calculator or it's really easy, or with a calculator otherwise. Don't be a hero by working difficult antiderivatives by hand when you have a calculator and permission to use it. First off, you're wasting valuable time. Secondly, you greatly increase your chances of error. And finally, you might actually be looking at a function with no nice antiderivative.[9] In that case, you wouldn't only be using up extra time — it would be for no reason!

[9] "Nice" = closed-form, which is to say, an ordinary function with no integral sign remaining.

Properties of definite integrals

Rule	Formula	Comments
Order of integration	$\int_a^b f(x)dx = -\int_b^a f(x)dx$	If you switch the limits of integration, you change the sign of the result. Thinking back to the areas of rectangles, it's like making the Δx values negative, so that the "areas" change sign.
Zero	$\int_a^a f(x)dx = 0$	If there's no base, there's no area.
Constant multiple	$\int_a^b kf(x)dx = k\int_a^b f(x)dx$	This works with $k = -1$, too, so a negative sign can be brought out. We use this idea all the time with u-substitution. Since k is a *constant*, only constants can be brought outside.
Sum and difference	$\int_a^b \big(f(x) \pm g(x)\big)dx$ $= \int_a^b f(x)dx \pm \int_a^b g(x)dx$	Just like with derivatives.
Additivity	$\int_a^b f(x)dx + \int_b^c f(x)dx$ $= \int_a^c f(x)dx$	Geometrically, this is easy to see if the values *a*, *b*, and *c* are left to right along the *x*-axis. The cool part is that it continues to work regardless of their relative sizes.

Integrating a function $f(x)$ [computing an antiderivative]

Fact: Integrals tend to be more difficult than derivatives, because you sometimes *can't* just look at them and know what rule to use. However, it helps enormously to be able to recognize common integrals. Those are the derivative rules in reverse.

Rule	Formula	Comments
Power	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	Again, this *only* works when the power is a number. The power goes up when you integrate: integrate is increase, differentiate is decrease.
Logarithm	$\int \frac{1}{x} dx = ln\lvert x\rvert + C$	The absolute value is essential. Only use this rule when the power rule fails! If the power on x is not -1, then this rule will not apply.
Exponential	$\int e^x dx = e^x + C$ $\int e^{kx} dx = \frac{1}{k} e^{kx} + C$	The second one is the chain rule with a constant coefficient on x.
Trigonometric	$\int \sin kx\, dx = -\cos x + C$ $\int \sin kx\, dx = -\frac{1}{k}\cos kx + C$	The derivative of sine is positive, so its *anti*derivative is negative, and so on.
	$\int \cos x\, dx = \sin x + C$ $\int \cos kx\, dx = \frac{1}{k}\sin kx + C$	
	$\int \sec^2 x\, dx = \tan x + C$ $\int \csc^2 x\, dx = -\cot x + C$	
	$\int \sec x \tan x\, dx = \sec x + C$ $\int \csc x \cot x\, dx = -\csc x + C$	
Inverse Trigonometric	$\int \frac{1}{1+x^2} dx = \arctan x + C$ $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$	

Integration by Substitution (u-substitution[10]):
If you can't find an antiderivative by "just looking," or using one of the rules above, then try u-substitution. It is about undoing the chain rule. Because of that, you're looking for something that corresponds to the "inside" function in a chain rule derivative.

Process for indefinite integrals:

0. For this example, I'm using the generic chain rule result as the integrand.	$\int f'(g(x))g'(x)dx$
1. Decide what to use for u, and write down what you've selected.	Let $u = g(x)$.
2. Immediately take the derivative of this, and call it du. Don't forget the dx.	Then $du = g'(x)dx$.
3. Substitute those into the integral you started with to make it an easier problem to integrate.	$\int f'(g(x))g'(x)dx$ $= \int f'(u)du$
4. Integrate that function.	$= f(u) + C$
5. Go back to the original variable by replacing u.	$= f(g(x)) + C$

Process for definite integrals:

0. The difference here is that there are limits of integration.	$\int_a^b f'(g(x))g'(x)dx$
1. Decide what to use for u, and write down what you've selected.	Let $u = g(x)$.
2. Immediately take the derivative of this, and call it du. Don't forget the dx.	Then $du = g'(x)dx$.
3a. Calculate the new limits of integration. You do this by taking the values of x, which are a and b, and substituting them into the formula you wrote for u. (Note that $g(a)$ and $g(b)$ are numbers.)	If $x = a$, then $u = g(a)$; if $x = b$, then $u = g(b)$.
3b. Substitute those into the integral you started with to make it an easier problem to integrate.	$\int_a^b f'(g(x))g'(x)dx$ $= \int_{g(a)}^{g(b)} f'(u)du$
4. Integrate that function.	$= [f(u)]_{g(a)}^{g(b)}$
5. Evaluate the definite integral using the limits in terms of u. It is *not* necessary to go back to x. Just finish the problem here.	$= f(g(b)) - f(g(a))$

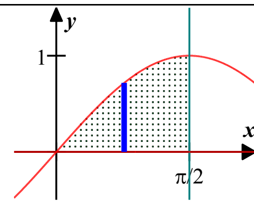
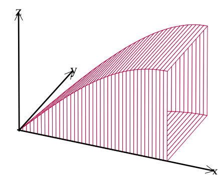
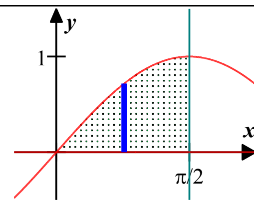
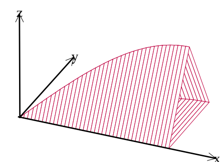
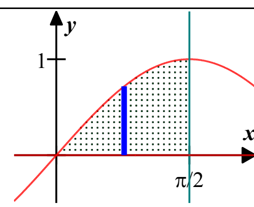
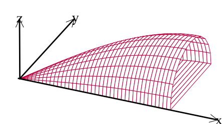
Be careful: The du NEVER goes in the denominator of the integrand, or under a radical, or inside a function, or anywhere but right at the end.

[10] Listen to the Soulja Boy song.

Finding volumes of solids

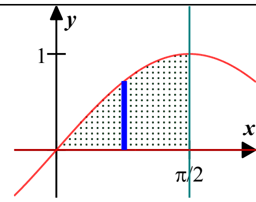
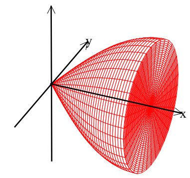
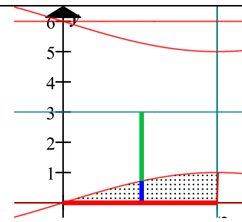
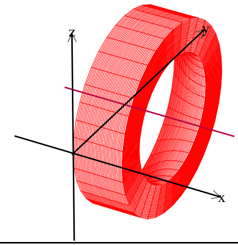
Fact: Just like you integrate the formula for a curve to find the associated area, you can integrate an area formula to find a volume.

Fact: If you have a three-dimensional object with cross sections that correspond to a geometric figure, integrate the area formula for that shape. Then you substitute appropriate information about the base and height from the curve itself in order to get a function you can integrate. In each case here, the base of the figure is the region bounded by $y = \sin x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$. The cross sections are perpendicular to the x-axis, hence the dx. In each case below, the height of the blue rectangle is y, the distance from the x-axis to the curve.

Cross sections	Formula	Integral	Base	Figure
Squares	$A = s^2$	$\int_0^{\pi/2} s^2 dx =$ $\int_0^{\pi/2} y^2 dx =$ $\int_0^{\pi/2} (\sin x)^2 dx$	 	
Equilateral triangles	$A = \frac{1}{2}bh =$ $\frac{1}{2}ab\sin C =$ $\frac{1}{2}bb\ sin\ 60° =$ $\frac{1}{2}b^2 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}b^2$	$\frac{\sqrt{3}}{4}\int_0^{\pi/2} b^2 dx$ $= \frac{\sqrt{3}}{4}\int_0^{\pi/2} y^2 dx$ $= \frac{\sqrt{3}}{4}\int_0^{\pi/2} (\sin x)^2 dx$	 	
Semicircles with diameters on the xy-plane	$A = \frac{1}{2}\pi r^2$, but be careful. In this case, r is *half* of y.	$\frac{1}{2}\pi\int_0^{\pi/2} r^2 dx$ $= \frac{1}{2}\pi\int_0^{\pi/2} \left(\frac{1}{2}y\right)^2 dx$ $= \frac{1}{2}\pi\int_0^{\pi/2} \left(\frac{1}{2}\sin x\right)^2 dx$ $= \frac{1}{8}\pi\int_0^{\pi/2} (\sin x)^2 dx$	 	

You'll probably notice how similar all of those look. It's not a coincidence.

Fact: To revolve the same region about the x-axis, you'd use the ***disc method***. This involves integrating the area of a circle, which looks a whole lot like those above. The object goes below the x-axis this time, though. Revolving it about a different horizontal line, $y = k$, involves using the ***washer method***, or as I like to think about it, discs with circles cut out. I'll use the line $y = 3$ in the example.

Cross sections	Formula	Integral	Base	Figure
Circles formed by revolving the region about the x-axis	$A = \pi r^2$	$\pi\int_0^{\pi/2} r^2 dx$ $= \pi\int_0^{\pi/2} y^2 dx$ $= \pi\int_0^{\pi/2} (\sin x)^2 dx$		
Washers formed by revolving the region about the horizontal line $y = 3$	$A = \pi R^2 - \pi r^2$	$\pi\int_0^{\pi/2} (R^2 - r^2)dx$ $= \pi\int_0^{\pi/2} (3^2 - \text{green}^2)dx$ $= \pi\int_0^{\pi/2} (3^2 - (3 - \sin x)^2)dx$		

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