Be Prepared Practice Exam BC-2
ONLY the AB-accessible problems

Multiple Choice

1. $f(x) = \dfrac{x^2}{\tan 2x}$

$$f'(x) = \frac{\tan 2x \cdot 2x - x^2 \cdot \sec^2 2x \cdot 2}{(\tan 2x)^2} = \frac{2x \tan 2x - 2x^2 \sec^2 2x}{\tan^2 2x} \quad \boxed{B}$$

2. I is a polynomial, so is both continuous and differentiable for all $x \in \mathbb{R}$: yes.
II has a corner at $x = 1.5$, and is not differentiable for $x \in (0,2)$, so no.
III has a discontinuity at $x=1$, so no. $\boxed{A}$

3. BC only

4. BC only

5. $\sin 2x + \cos 2y = x - y$

$$\cos 2x \cdot 2 + -\sin 2y \cdot 2 \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - 2\sin 2y \frac{dy}{dx} = 1 - 2\cos 2x$$

$$\frac{dy}{dx}(1 - 2\sin 2y) = 1 - 2\cos 2x$$

$$\frac{dy}{dx} = \frac{1 - 2\cos 2x}{1 - 2\sin 2y} \quad \boxed{A}$$

6. $\lim_{w\to 0}\left(\dfrac{\ln\left(\frac{2+w}{2}\right)}{w}\right) = \dfrac{\ln 1}{0} = \dfrac{0}{0}$ L'Hôpital:

$= \lim_{w\to 0} \dfrac{\frac{2}{2+w}\cdot\frac{2\cdot 1-(2+w)0}{4}}{1} = \lim_{w\to 0}\left(\frac{2}{2+w}\cdot\frac{1}{2}\right) = \frac{2}{2+0}\cdot\frac{1}{2} = 1\cdot\frac{1}{2} = \frac{1}{2}$ $\boxed{B}$

7. $g(x) = \sin 2x + x^2$, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$g'(x) = 2\cos 2x + 2x$

$g''(x) = 2\cdot -2\sin 2x + 2 = -4\sin 2x + 2$

$g''(x) = 0 = -4\sin 2x + 2$

$4\sin 2x = 2$

$\sin 2x = \frac{1}{2}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow x = \frac{\pi}{12}, \cancel{\frac{5\pi}{12}} \leftarrow$ bigger than $\frac{\pi}{4}$

And g'' will change signs when $x=0$ $\boxed{C}$

8. I'll examine each possibility in order. If this were an AP exam, I believe the x- and y-axes would be drawn in for you.

A: $\frac{dy}{dx} = -\frac{x}{y}$. Not possible, because this would be undefined when $y=0$, and there are horizontal tangents near the origin where $y=0$.

B: All of the slopes shown appear to be positive or zero. Since $\frac{dy}{dx} = \arctan x$ will give negative answers for $x<0$, it's not right.

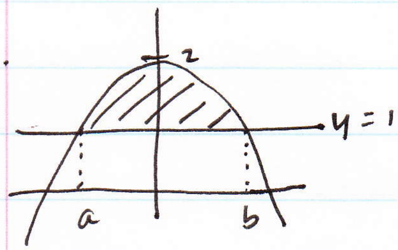
C: $\frac{dy}{dx} = x^3$ ALSO gives negatives for $x<0$. Nope

D: $\frac{dy}{dx} = x+y$ will be negative in the third quadrant and then some. So no.

E: $\frac{dy}{dx} = x^2+y^2$ is always nonnegative. The further away from $(0,0)$, the steeper the slopes get. Yep $\boxed{E}$

9. $F'(x) = \sec^2 3x$. So F will have started out as a tangent. This narrows down to A, B, and C. Because I can envision the u-substitution, I know the answer is C, but I'll check it.

$$\frac{d}{dx}\left(\frac{1}{3}\tan 3x\right) = \frac{1}{3}\cdot \sec^2 3x \cdot 3 = 1\cdot \sec^2 3x.$$ yep. $\boxed{C}$

10. 

I'll integrate top minus bottom, but first I need the values of a and b there.

$$2\cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{-\pi}{3}$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((2\cos x) - 1\right) dx = \left[2\sin x - x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$
$$= \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(2\sin\left(\frac{-\pi}{3}\right) - \left(\frac{-\pi}{3}\right)\right)$$
$$= \left(2\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right) - \left(2\cdot\frac{-\sqrt{3}}{2} + \frac{\pi}{3}\right)$$
$$= \sqrt{3} - \frac{\pi}{3} + \sqrt{3} - \frac{\pi}{3} = 2\sqrt{3} - \frac{2\pi}{3}$$ $\boxed{D}$

11. BC only

12. This is a disguised average value of a function problem. The average length of the segments is the average value of the function $f(x) = \tan x - \sin x$ the coefficients on the answers are also a clue.

$$\frac{1}{\frac{\pi}{3} - \frac{\pi}{4}} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\tan x - \sin x) = \frac{12}{\pi}\Big[\ln|\sec x| + \cos x\Big]_{\pi/4}^{\pi/3}$$
$$= \frac{12}{\pi}\left[\left(\ln\left|\sec\frac{\pi}{3}\right| + \cos\frac{\pi}{3}\right) - \left(\ln\left|\sec\frac{\pi}{4}\right| + \cos\frac{\pi}{4}\right)\right] = \frac{12}{\pi}\left[\left(\ln 2 + \frac{1}{2}\right) - \left(\ln\sqrt{2} + \frac{1}{\sqrt{2}}\right)\right]$$
$$= \frac{12}{\pi}\left[\ln\frac{2}{\sqrt{2}} + \frac{1}{2} - \frac{\sqrt{2}}{2}\right] = \frac{12}{\pi}\left(\ln\sqrt{2} + \frac{1-\sqrt{2}}{2}\right)$$ $\boxed{A}$

13. If $f'(x) = -x^4$, $f' \leq 0$ for all x, so yes, decr. on $[1,2]$

If $g'(x) = \sin(\pi x)$, then on $[1,2]$, we're between $\sin \pi$ and $\sin 2\pi$. On the unit circle, this is quadrants III and IV, where sine is negative. So decreasing.

If $h'(x) = 3 - 2x$, then $h'(1) = 3 - 2 = 1$, not negative. Nope. $\boxed{D}$

14. BC only

15. $\frac{dy}{dx} = -0.5x \Rightarrow \int dy = \int -0.5x\,dx$

$$y = \frac{-0.5x^2}{2} = \frac{-1}{4}x^2 + C \qquad \boxed{E}$$

16. $g(x) = \lim\limits_{h \to 0} \dfrac{\tan(x+h) - \tan x}{h}$.

This is a derivative formula, and $g(x) =$ the derivative of $\tan x$.

So $g(x) = \sec^2 x$.

Then $g'(x) = 2\sec x \cdot \sec x \tan x$.

$$g'\left(\tfrac{\pi}{3}\right) = 2\sec\tfrac{\pi}{3} \cdot \sec\tfrac{\pi}{3} \cdot \tan\tfrac{\pi}{3} = 2 \cdot 2 \cdot 2 \cdot \sqrt{3} = 8\sqrt{3} \qquad \boxed{A}$$

17. BC only

18. $\frac{dy}{dx} = \frac{2x+1}{3y^2}$ $y=2$ when $x=1$

Find y when $x=3$.

$\int 3y^2\,dy = \int (2x+1)\,dx$

$y^3 = x^2+x+C$

$2^3 = 1^2+1+C$

$8 = 2+C$

$C = 6$

So $y^3 = x^2+x+6$, and

$y = \sqrt[3]{x^2+x+6}$.

When $x=3$, $y = \sqrt[3]{3^2+3+6}$

$= \sqrt[3]{9+9} = \sqrt[3]{18}$ $\boxed{E}$

19. BC only

20. $f(x) = \begin{cases} \ln 4 - \frac{x}{4}, & \text{if } x \le 4 \\ \ln x - \frac{x^2}{16}, & \text{if } x > 4 \end{cases}$

I. $\lim_{x\to 4^-} f(x) = \ln 4 - \frac{4}{4} = \ln 4 - 1$

$\lim_{x\to 4^+} f(x) = \ln 4 - \frac{4^2}{16} = \ln 4 - 1$ Looks continuous. True

II. $f'(x) = \begin{cases} -\frac{1}{4}, & \text{if } x < 4 \\ \frac{1}{x} - \frac{1}{16}\cdot 2x, & \text{if } x > 4 \end{cases}$

The only potential problems with the derivative existing are at the value where the pieces meet and at $x=0$. Since the second line is only for $x>4$, $x=0$ isn't a problem.

$\lim_{x\to 4^-} f'(x) = \frac{-1}{4}$ $\lim_{x\to 4^+} f'(x) = \frac{1}{4} - \frac{1}{16}\cdot 8 = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4}$.

So I can put $\le$ on the first part of f', and II is true

(continued)

20. (continued)

III: Since we already know that f is continuous, it must be integrable. True. $\boxed{E}$

21. BC only. The integral requires partial fractions.

22. $g(x) = 6 + \int_{-2}^{x} \cos(w^5)\,dw$

$g'(x) = \cos(x^5)$ FTC, straight up. $\boxed{C}$

23. BC only, requires integration by parts

24. $g(x) = \int_{0}^{x} f(t)\,dt - \int_{2}^{x} f(t)\,dt$

$g'(x) = f(x) - f(x) = 0$. So $g''(x) = 0$, too.

Since g'' never changes signs, there can be no points of inflection. $\boxed{A}$

25. $\lim\limits_{x\to\infty} \dfrac{\arctan x + 2x}{x + e^{-x}} = \dfrac{\frac{\pi}{2} + \infty}{\infty + 0} = \dfrac{\infty}{\infty}$. L'Hôpital:

$\lim\limits_{x\to\infty} \dfrac{\frac{1}{1+x^2} + 2}{1 + -e^{-x}} = \dfrac{0+2}{1+0} = 2$ $\boxed{D}$

26. BC only

27. BC only

28. BC only

29. $g(x) = \int_1^x f(t)\,dt$

$g(0) = \int_1^0 f(t)\,dt$. Since the interval goes right to left, and $f(x) > 0$, $g(0)$ must be negative, eliminating choices A and E.

$g(1) = \int_1^1 f(t)\,dt = 0$, eliminating choice B.

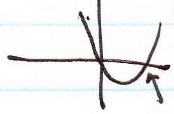
The difference between C and D is the relative size of the areas. Since the area between 0 and 1 is larger than that from 1 to 2, the answer must be $\boxed{C}$.

30. $f'(x) < 0$. $\quad g(x) = f(x^3 + e^{-x^2})$

$g'(x) = f'(x^3 + e^{-x^2}) \cdot (3x^2 + -2xe^{-x^2})$

As $f'(x) < 0$, g' will change from + to − when $3x^2 - 2xe^{-x^2}$ changes from − to +. I will graph $y = 3x^2 - 2xe^{-x^2}$.

$y = 3x^2 - 2xe^{-x2}$



$x \approx 0.513$

$\boxed{E}$

31. $y = x^3 - e^x$ for $x \in [0,2]$.

Instantaneous r.o.c. is $\frac{dy}{dx} = 3x^2 - e^x$

Average r.o.c. $= \frac{f(b)-f(a)}{b-a} = \frac{(2^3-e^2)-(0^3-e^0)}{2-0} = \frac{9-e^2}{2}$

Solving $3x^2 - e^x = \frac{9-e^2}{2}$ by graphing gives $x \approx -0.663$, 3.690, and 1.149.

Only one of these is on $[0,2]$ $\boxed{D}$

32. Functions with the same derivative must differ at most by a constant. This makes them vertical translations of each other. $\boxed{A}$

33. $\lim_{x\to 0} \frac{\sin 3x + ax + bx^3}{x^3} = \frac{0}{0}$ L'Hopital.

$\lim_{x\to 0} \frac{3\cos(3x) + a + 3bx^2}{3x^2}$ will have 0 in denominator. If it is to be defined as $x \to 0$, we'd need L'Hopital to apply again.

That means $3\cos 0 + a + 3b \cdot 0^2 = 0$

$3 + a = 0$, and $\boxed{a = -3}$

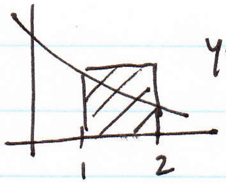
So now $\lim_{x\to 0} \frac{3\cos(3x) + -3 + 3bx^2}{3x^2} = \lim_{x\to 0} \frac{-9\sin(3x) + 6bx}{6x}$

Same situation, and I need $-9\sin 0 + 6b \cdot 0 = 0$, which is already true. L'Hô one more time.

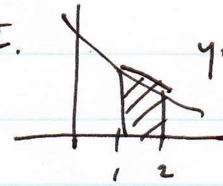
$\lim_{x\to 0} \frac{-27\cos(3x) + 6b}{6} = 0$ only if $-27\cos(0) + 6b = 0$

$-27 + 6b = 0$

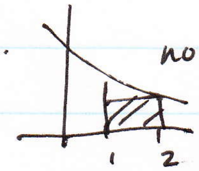
$\boxed{b = \frac{27}{6} = \frac{9}{2}}$ $\boxed{E}$

34. I. 

yes

II. 

yes

III. 

no.

$\boxed{C}$

35. BC only

36. Riemann sum is areas of rectangles. Four subintervals means the endpoints are at 0, 2, 4, 6, 8.

$$\sum(\text{base}\cdot\text{height}) = 2\cdot 130 + 2\cdot 118 + 2\cdot 108 + 2\cdot 120 = 952 \qquad \boxed{E}$$

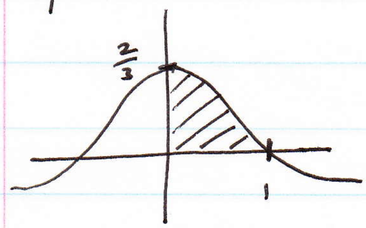
37. BC only

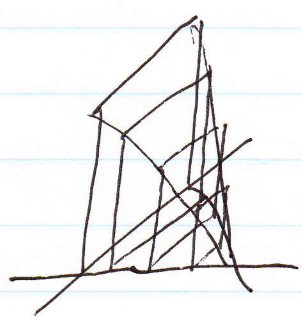
38. $m'(x) = \cos(1-x^2)$; $m(2) = 1$

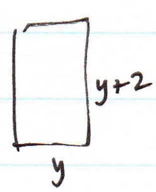
$$m(3) = m(2) + \int_2^3 m'(x)\,dx \approx 1.104 \qquad \boxed{D}$$

39. BC only

40. $y = 3^{-x^2} - \frac{1}{3}$







$$\int_0^1 y(y+2)\,dx = \int_0^1 \left(3^{-x^2}-\tfrac{1}{3}\right)\left(3^{-x^2}-\tfrac{1}{3}+2\right)dx \approx 0.992 \qquad \boxed{D}$$

41. $F'(x) = G(x)$

So $\int G(x)\,dx = F(x) + C$

$\int_3^5 G(x)\,dx = F(5) - F(3) = 4 - (-2) = 6$ $\boxed{C}$

42. $f(x) = \int_2^x g(t)\,dt$. $f(2) = 0$ for obvious reasons, but if

$f(1)$ also is 0, then the areas above and below the x-axis must "cancel out" on $[1, 2]$. That eliminates D and E.

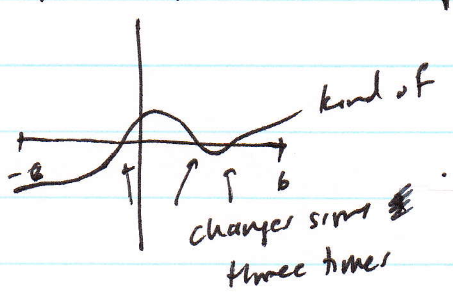
~~Since~~ $\cancel{f(0) = \int_2^0 g(t)\,dt > 0}$ no help, they all look the same on $[0, 1.5]$.

Since $f(3) = \int_2^3 g(t)\,dt = 1.3$, which is greater than 1,

but $f(4)$ increases by less than 1.3 from there, the answer must be A; the area from 2 to 3 is more than that from 3 to 4. $\boxed{A}$

43. $f''(x) = 2^{-\frac{x^2}{5}} \cdot \cos x + \frac{x}{6} - 0.1$.

Points of inflection happen when f'' changes signs. A graph:



$\boxed{C}$

44. ~~h~~ $h(x) = f^{-1}(x)$. $h(2)$ is the solution to $f(x)=2 \Rightarrow h(2)=-1$

$$h'(2) = (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(h(2))} = \frac{1}{f'(-1)} = \frac{1}{-1/2} = -2 \qquad \boxed{B}$$

45. This seems really easy for a "last question". The slopes of h are all positive, and as you move from left to right they get smaller (closer to 0). Looks like $\boxed{C}$.

Free Response

1. $f(x) = \dfrac{4\cos x - x}{2 + \sqrt[3]{x^2}}$; $g'(x) = f(x)$; $h'(x) = \int_0^x f(t)\,dt$; $g(0) = 1$.

a) $g(0) = 1$, so the point is $(0,1)$.

$$g'(0) = f(0) = \frac{4\cos 0 - 0}{2 + \sqrt[3]{0^2}} = \frac{4}{2} = 2 = \text{slope}$$

$$\boxed{y - 1 = 2(x-0)}$$

b) We're <u>only</u> interested in the interval $[0,1]$. On that interval, $f(x) = g'(x)$ is strictly positive so g is increasing. Its maximum will occur at the right end point.

$$g(1) = g(0) + \int_0^1 g'(x)\,dx = 1 + 1.14619 = \boxed{2.146}$$

(continues)

1. (continued)

c) h will have any points of inflection when h'' changes signs. $h'(x) = \int_0^x f(t)\,dt$, so $h''(x) = f(x)$.

In the graph, f(x) changes signs at all three of its zeros on the interval. Solving on the calculator,

$f(x) = 0 \Rightarrow \boxed{x \approx -3.595,\ -2.133,\ 1.252}$

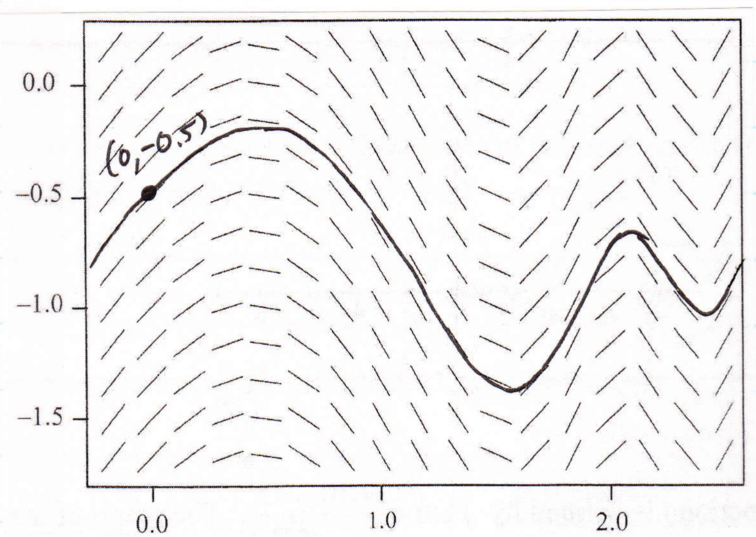
d) $h'(2) = \int_0^2 f(t)\,dt \approx 0.795$

$h'(3) = \int_0^3 f(t)\,dt = -0.649$

So $\boxed{h'(2) \text{ is larger.}}$

2. a) BC only

b) $\frac{dy}{dx} = 2\cos(e^x)$, and $y(0) = -0.5$



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2. (continued)

c) I can't integrate that. Using the FTC here mean the answer will involve an integral.

$$\boxed{y(x) = y(0) + \int_0^x 2\cos(e^t)\,dt} = -0.5 + \int_0^x 2\cos(e^t)\,dt$$

$$y(1) = y(0) + \int_0^1 2\cos(e^t)\,dt = \boxed{-0.747}$$

d) $\frac{d^2y}{dx^2} = -2e^x \sin(e^x)$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} \approx -2.233$, so the graph of y is concave down when $x=1$. The tangent line (which would have given the approximation in (a) that you didn't find) would be <u>above</u> the curve, so its value is greater than the true value in part (c)

3. $g(x) = \int_4^x h(t)\,dt$

a) Point $(6, g(6))$.

$g(6) = \int_4^6 h(t)\,dt = \frac{1}{2}\cdot 2\cdot 2 = 2$, so point is $(6,2)$.

Slope is $g'(6)$

Since $g(x) = \int_4^x h(t)\,dt$, $g'(x) = h(x)$.

Then $g'(6) = h(6) = 2$, from the graph. Slope is 2.

tangent line: $\boxed{y - 2 = 2(x-6)}$

b) $g'(x) = h(x)$. On the interval $[-4, -3]$, $h(x) \geq 0$, so $g(x)$ is increasing. Therefore $\boxed{g(-3)} > g(-4)$.

c) The absolute maximum of g can occur at an end point or at a critical point where g' changes signs from positive to negative. The left endpoint is a possibility, since to the right of -7, $g' < 0$, and g is decreasing. However, as $\int_{-4}^{0} g'(x)\,dx > \left|\int_{-7}^{-4} g'(x)\,dx\right|$,

$g(-7)$ will not be the maximum. The right endpoint is not a possibility, because $g'(x) \leq 0$ on $x \in [7, 8]$. The maximum must occur when $g'(x)$ changes from positive to negative, at $x = 7$. $g(7) = \int_4^7 h(t)\,dt = \frac{1}{2}\cdot 3\cdot 2 = \boxed{3}$

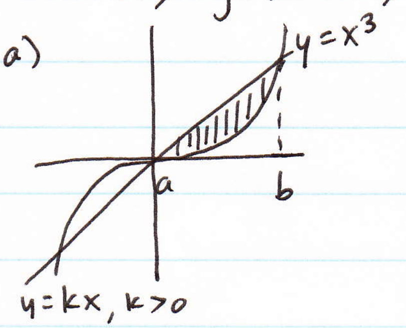
(continues)

3. (continued)

d) $g(x)$ will have a point of inflection where g'' changes signs. Since $g'(x) = h(x)$, then $g''(x) = h'(x)$, and g'' changes sign when h has a relative maximum or minimum on $x \in (-7, 8)$. This happens at $\boxed{x = 0}$, $\boxed{x = 4}$, and at $\boxed{x = 6.}$

4. $f(x) = x^3$, $g(x) = kx$, $\frac{dk}{dt} = 9$, $k > 0$

a)



Area of $R = \int_a^b (kx - x^3)\,dx$

Clearly $x^3 = x \cdot k$ when $x = 0$, so $a = 0$.

But $x^3 - kx = 0$

$x(x^2 - k) = 0$

$x = 0$ or $x = \pm\sqrt{k}$, and $b = \sqrt{k}$

$$\text{Area} = \int_0^{\sqrt{k}} (kx - x^3)\,dx = \left[\frac{1}{2}kx^2 - \frac{1}{4}x^4\right]_0^{\sqrt{k}}$$

$$= \left(\frac{1}{2}k \cdot (\sqrt{k})^2 - \frac{1}{4}(\sqrt{k})^4\right) - (0 - 0) = \frac{1}{2}k \cdot k - \frac{1}{4} \cdot k^2$$

$$= \frac{1}{2}k^2 - \frac{1}{4}k^2 = \boxed{\frac{1}{4}k^2}$$

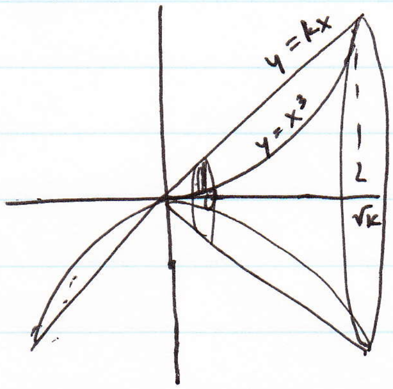
b) $A = \frac{1}{4}k^2$

$$\frac{dA}{dt} = \frac{2}{4}k\,\frac{dk}{dt} = \frac{2}{\cancel{4}} \cdot \cancel{4} \cdot 9 = \boxed{18 \text{ units}^2/\text{sec}}$$

(continued)

4. (continued)

c)



$$\pi \int_a^b (R^2 - r^2)\,dx$$

$$= \pi \int_0^{\sqrt{k}} \left((kx)^2 - (x^3)^2\right) dx$$

$$= \pi \int_0^{\sqrt{k}} \left(k^2x^2 - x^6\right) dx$$

$$= \pi \left[\frac{1}{3}k^2x^3 - \frac{1}{7}x^7 \right]_0^{\sqrt{k}} = \pi \left[\left(\frac{1}{3}k^2(\sqrt{k})^3 - \frac{1}{7}(\sqrt{k})^7 \right) - (0) \right]$$

$$= \pi \left(\frac{1}{3} \cdot k^2 \cdot k\sqrt{k} - \frac{1}{7} \cdot k^3\sqrt{k} \right) = \pi \left(\frac{1}{3}k^3\sqrt{k} - \frac{1}{7}k^3\sqrt{k} \right)$$

$$= \boxed{\frac{4}{21}\pi k^3\sqrt{k}}$$

d) $V = \frac{4}{21}\pi k^3\sqrt{k} = \frac{4}{21}\pi k^{\frac{7}{2}}$

$$\frac{dV}{dt} = \frac{\cancel{7}}{\cancel{2}} \cdot \frac{\cancelto{2}{4}}{\cancelto{3}{21}}\pi k^{\frac{5}{2}} \cdot \frac{dk}{dt} = \frac{2}{3}\pi \cdot 4^{5/2} \cdot 9 = \frac{2}{\cancel{3}} \cdot \pi \cdot 32 \cdot \cancelto{3}{9}$$

$$= 64\pi \cdot 3 = \boxed{192\pi \text{ units}^3/\text{second}}$$

5. BC only

6. BC only