Be Prepared Practice Exam BC-1

ONLY the AB-accessible questions

1. Find $(f^{-1})'(\frac{1}{2})$.

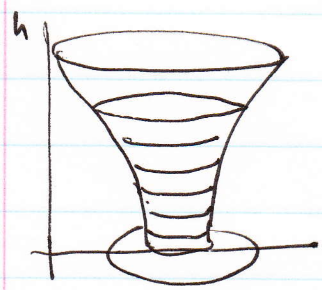
The formula is $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

So I need $(f^{-1})'(\frac{1}{2}) = \dfrac{1}{f'(f^{-1}(\frac{1}{2}))}$.

Since $f(\frac{3}{2}) = \frac{1}{2}$, $f^{-1}(\frac{1}{2}) = \frac{3}{2}$.

Then $(f^{-1})'(\frac{1}{2}) = \dfrac{1}{f'(\frac{3}{2})} = \dfrac{1}{-\frac{1}{2}} = -2$ $\boxed{B}$

2. $\int_0^2 f'(x)\,dx = f(x)\Big]_0^2 = f(2) - f(0) = \frac{3}{2} - 1 = \frac{1}{2}$ $\boxed{D}$

3. 

As h gets bigger, obviously there's more volume. But initially, a small change in height adds a ~~cylinder~~ disc of water with a small radius, like  that. The higher the water gets, the bigger those discs get, like  and then . So volume increases faster as we go up inch by inch. that gives a graph like $\boxed{A}$

4. BC only

5. $f(x)$ is continuous and even.

$$\int_0^4 f(x)\,dx = -5 \qquad \int_4^6 f(t)\,dt = 2$$

Avg. value of f over $[-6, 4]$ $= \frac{1}{4-\text{-}6}\int_{-6}^{4} f(x)\,dx$

Because f is even $\int_{-6}^{-4} f(t)\,dt = \int_4^6 f(t)\,dt = 2$ and

$\int_{-4}^{0} f(x)\,dx = \int_0^4 f(x)\,dx = -5.$ Whether you use x or

t makes no difference, so putting it all together,

$$\frac{1}{10}\int_{-6}^{4} f(x)\,dx = \frac{1}{10}\left(\int_{-6}^{-4} f(x)\,dx + \int_{-4}^{0} f(x)\,dx + \int_0^4 f(x)\,dx\right)$$

$$= \frac{1}{10}\left(2 + -5 + -5\right) = \frac{-8}{10} \qquad \boxed{B}$$

6. Checking the answers is the only way I see to do this.

A: $f(0) = 1$ B: $f'(0) \approx -1$ C: $\lim\limits_{h\to 0} \frac{f(h)-1}{h} = \lim\limits_{h\to 0} \frac{f(h)-f(0)}{h-0}$

$= f'(0) \approx -1$ again

D: $\frac{f(1)-f(-1)}{2} = \frac{0-2}{2} = -1$ E: $\frac{f'(1)-f'(-1)}{2} = \frac{0-0}{2} = 0$

So the largest value is choice $\boxed{A}$

7. $\lim_{h\to 0}\left(\frac{1}{h}\int_1^{1+h} e^{-t^2}\,dt\right) = \lim_{h\to 0}\left(\frac{\int_1^{1+h} e^{-t^2}\,dt}{h}\right) = \frac{0}{0}$

So maybe L'Hôpital's rule...

$= \lim_{h\to 0}\frac{e^{-(1+h)^2}}{1}$ (by the FTC)

$= \frac{e^{-1}}{1} = \frac{1}{e}$, $\boxed{D}$

8. BC only; to solve the differential equation requires partial fractions

9. BC only

10. $\int \frac{\sqrt{x}}{\sqrt{x}+1}\,dx$ Let $u = \sqrt{x}$, so $du = \frac{1}{2}x^{-1/2}\,dx$

$du = \frac{1}{2\sqrt{x}}\,dx$

Or, equivalently, $dx = 2\sqrt{x}\,du$

$= \int \frac{u}{u+1}\cdot 2\sqrt{x}\,du = \int \frac{u}{u+1}\cdot 2u\,du = \int \frac{2u^2}{u+1}\,du$

$\boxed{E}$

11. $f(x) = \tan^{-1}(x) = \arctan(x)$

$f'(x) = \dfrac{1}{1+x^2}$

$$\lim_{x\to\sqrt{3}} \frac{f'(x) - f'(\sqrt{3})}{x-\sqrt{3}} = \lim_{x\to\sqrt{3}} \frac{\frac{1}{1+x^2} - \frac{1}{1+3}}{x-\sqrt{3}} = \frac{\frac{1}{4}-\frac{1}{4}}{\sqrt{3}-\sqrt{3}} = \frac{0}{0}$$

L'Hôpital: $\displaystyle \lim_{x\to\sqrt{3}} \frac{\frac{(1+x^2)\cdot 0 - 1(2x)}{(1+x^2)^2} - 0}{1-0} = \lim_{x\to\sqrt{3}} \frac{-2x}{(1+x^2)^2}$

$$= \frac{-2\sqrt{3}}{(1+3)^2} = \frac{-2\sqrt{3}}{16} = \frac{-\sqrt{3}}{8} \qquad \boxed{A}$$

12. If f is continuous at $x=2$, then

$\displaystyle \lim_{x\to 2} \frac{|x|-2}{x-2} = k.$ Since $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$,

$$\frac{|x|-2}{x-2} = \begin{cases} \dfrac{x-2}{x-2}, & x \geq 0 \\[2ex] \dfrac{-x-2}{x-2}, & x < 0 \end{cases}$$

$\cancel{\lim_{x\to 2} \frac{|x|+2}{x-2} = \frac{2+2}{2-2} = \frac{+4}{0}}$ ~~undefined~~

Since "$\lim\limits_{x\to 2}$" means x is near 2, we only need the first line of this definition.

$$\frac{x-2}{x-2} = 1, \text{ so } k = 1 \qquad \boxed{D}$$

13. BC only

14. The slopes are positive for $x > 0$, and are larger when x is close to 0. The graph is symmetric about the y-axis, so slopes are negative for $x < 0$.

Eliminate B, because all slopes would be positive.

Eliminate D, because $\ln x$ is undefined for $x < 0$.

Eliminate E, because the slopes show no periodicity, and would not be the same sign on one side

Since the slope does not change with y (and isn't 0 when $y=0$), eliminate C.

So the answer is $\boxed{A}$

15. No calculator! So asymptotes come from limits.

$$f(x) = \frac{\sin x}{|x|}$$

Test $x=0$, which looks like it could be a vertical asymptote:

$\lim\limits_{x\to 0} \frac{\sin x}{|x|} = \frac{0}{0}$. If $x \geq 0$, $|x| = x$, so

$$\lim_{x\to 0^+} \frac{\sin x}{|x|} = \lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\cos x}{1} = 1, \text{ not } \infty.$$

If $x < 0$, $|x| = -x$, so $\lim\limits_{x\to 0^-} \frac{\sin x}{|x|} = \lim\limits_{x\to 0^-} \frac{\sin x}{x} = \lim\limits_{x\to 0^-} \frac{\cos x}{x} = 1$, not ∞.

No vertical asymptotes.

For horizontal asymptotes, examine $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

$$\lim_{x\to\infty} \frac{\sin x}{|x|} = \frac{\text{relatively small}}{\text{enormous}} = 0$$

$$\lim_{x\to-\infty} \frac{\sin x}{|x|} = \frac{\text{relatively small}}{\text{enormous}} = 0$$, so one horiz. asymptote at $y=0$

$\boxed{B}$

16. $f(x) = 4^{3x}$ Rule is $\frac{d}{dx}[a^x] = a^x \ln a$

$f'(x) = 4^{3x} \cdot \ln 4 \cdot 3$ $\boxed{C}$

17. BC only

18. BC only

19. $f(x) = e^x$

$f(f(x)) = e^{e^x}$

$\frac{d}{dx}\left[e^{e^x}\right] = \underbrace{e^{e^x} \cdot \frac{d}{dx}(e^x)}_{\text{chainrule}} = \underbrace{e^{e^x} \cdot e^x}_{\text{add the exponents}} = e^{e^x + x}$ $\boxed{E}$

20. BC only

21. $\frac{d}{dx}\int_x^{2x} \ln t \, dt = \frac{d}{dx}\left(\int_x^1 \ln t \, dt + \int_1^{2x} \ln t \, dt\right)$

$= \frac{d}{dx}\left(-\int_1^x \ln t \, dt + \int_1^{2x} \ln t \, dt\right)$

$= -\ln x + \ln 2x \cdot 2$

$= 2\ln 2x - \ln x = \ln(2x)^2 - \ln x$

$= \ln\left(\frac{4x^2}{x}\right) = \ln(4x)$ $\boxed{E}$

22. BC only

23. BC only (because there's no upper limit)

24. BC only

25. $$\int_0^2 x\sqrt{4-x^2}\,dx$$

Let $u = 4-x^2$. Then $du = -2x\,dx$. $x=0 \Rightarrow u=4$; $x=2 \Rightarrow u=0$

$$-\frac{1}{2}\int_0^2 -2x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_4^0 u^{1/2}\,du$$

$$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_4^0 = -\frac{1}{3}\left(0^{3/2}-4^{3/2}\right) = -\frac{1}{3}\cdot(0-8) = \frac{8}{3} \quad \boxed{C}$$

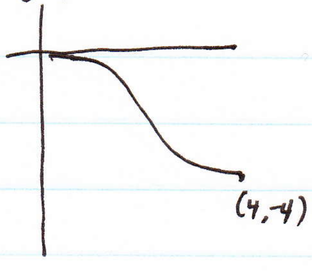
26. BC only

27. $f(x)$ increases when $f'>0$, and most rapidly when f' is as large as possible, at c_4 $\boxed{D}$

28. BC only

29. $F(x) = \int_x^0 f(t)\,dt = -\int_0^x f(t)\,dt$. Therefore $F'(x) = -f(x)$.

The graph of $F'(x)$ looks like



So F decreases always, slope of 0 at $x=0$, steepest at $x=4$.

$\boxed{D}$

30. $f(x) = \int_0^x \cos(t^2)\,dt$

$f'(x) = \cos(x^2)$. Tangent line at $x = \sqrt{\pi}$ has

slope $= f'(\sqrt{\pi}) = \cos\pi = -1$

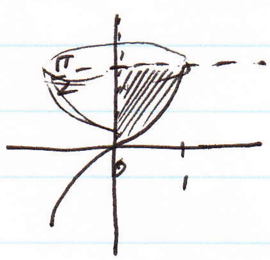
$f(\sqrt{\pi}) = \int_0^{\sqrt{\pi}} \cos(t^2)\,dt \approx 0.663$ (calculator)

Tangent line is $y - 0.663 = -1(x - \sqrt{\pi})$

When $x = 2$, then, $y - 0.663 = -1(2 - \sqrt{\pi})$

$y = 0.435$ $\boxed{C}$

31. $y = \sin^{-1}x = \arcsin x$, $y = \frac{\pi}{2}$, $x = 0$



Shells: Volume $= 2\pi \int_0^1 rh\,dx$

$= 2\pi \int_0^1 x\left(\frac{\pi}{2} - \arcsin x\right)dx \approx 2.4674$

$\boxed{A}$

→ 32. If f is differentiable for $x > 0$, then...

I. Yes, it's continuous for $x > 0$

II. Yes, since $x > 0$, $-x < 0$, so $f(-x)$ is differentiable on the negative numbers.

III. No. Tricky. The absolute value makes any negative x-value positive, but it does NOT address what happens at $x = 0$. Could be a corner or discontinuity, etc.

$\boxed{A}$

33. Let p = price. Then $\frac{dp}{dt} = k\sqrt{t}$, where t is measured in weeks.

$p(0) = 5$, $p(1) = 5.05$, find $p(12)$.

Separate the variables: $dp = k\sqrt{t}\,dt$

$$\int dp = \int k\sqrt{t}\,dt = \int kt^{1/2}dt$$

$$p = \frac{2}{3}kt^{3/2} + C$$

$p(0) = 5$: $\quad 5 = \frac{2}{3}k \cdot 0^{3/2} + C = 0 + C \Rightarrow C = 5$, so $p(t) = \frac{2}{3}kt^{3/2} + 5$

$p(1) = 5.05$: $\quad 5.05 = \frac{2}{3} \cdot k \cdot 1^{3/2} + 5 = \frac{2}{3}k + 5$

$$0.05 = \frac{2}{3}k, \text{ so } k = \frac{3}{2} \cdot 0.05 = 0.075$$

And therefore $p(t) = \frac{2}{3} \cdot 0.075\, t^{3/2} + 5$

$$p(t) = 0.05t^{3/2} + 5$$

$p(12) = 0.05 \cdot 12^{3/2} + 5 \approx \$7.08$ $\quad \boxed{D}$

34. BC only

35. BC only

36. $d = \sqrt{3} \cdot s$; $\quad V = s^3$; $\quad \frac{dd}{dt} = 0.5$ cm/s; $\quad d = 3$ cm. $\quad$ Find $\frac{dV}{dt}$

$V = s^3$ $\quad\quad d = \sqrt{3}s$ $\quad\quad d = 3 = \sqrt{3}s$

$\frac{dV}{dt} = 3s^2\frac{ds}{dt}$ $\quad\quad \frac{dd}{dt} = \sqrt{3} \cdot \frac{ds}{dt}$ $\quad\quad s = \frac{3}{\sqrt{3}} = \sqrt{3}$

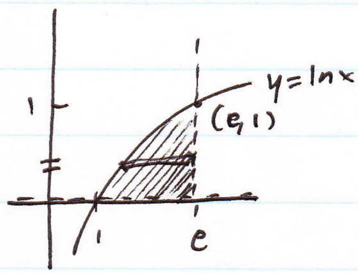
$\left.\frac{dV}{dt}\right|_{s=\sqrt{3}} = 3 \cdot (\sqrt{3})^2 \cdot \frac{0.5}{\sqrt{3}}$ $\quad\quad 0.5 = \sqrt{3} \cdot \frac{ds}{dt}$

$\quad\quad \frac{ds}{dt} = \frac{0.5}{\sqrt{3}}$

$= \frac{3 \cdot 3 \cdot 0.5}{\sqrt{3}} = \frac{3}{2}\sqrt{3} \approx 2.598$ $\quad \boxed{B}$

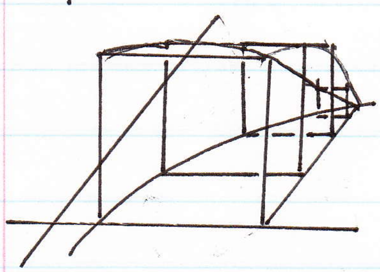
37. BC only

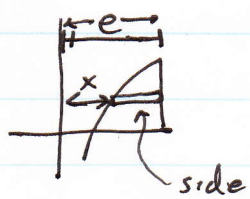
38.



Because the cross-sections are $\perp$ to the y-axis, it will be dy

$$\int_0^1 \begin{pmatrix}\text{area of}\\ \text{square}\end{pmatrix} dy = \int_0^1 (\text{side}^2)\, dy$$





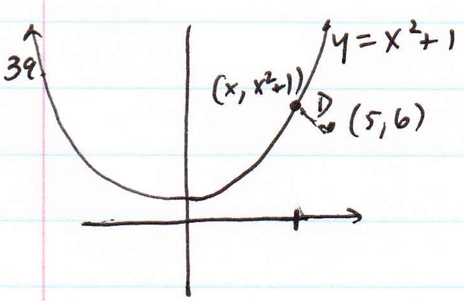
side $= e - x$.

We need this in terms of y, though.

$y = \ln x \Rightarrow x = e^y$. So side $= e - e^y$.

$$\text{Volume} = \int_0^1 (e - e^y)^2\, dy \qquad \boxed{A}$$

39.



Distance $= D = \sqrt{(x-5)^2 + (x^2+1-6)^2}$

Since you have a calculator, you could get away with just graphing this to find the minimum. I won't.

Let $r = (x-5)^2 + (x^2-5)^2$, the radicand. Finding the x that minimizes the radicand will also minimize the distance.

$$\frac{dr}{dx} = 2(x-5)' \cdot 1 + 2(x^2-5)' \cdot 2x = 2x - 10 + 4x^3 - 20x$$

$$= 4x^3 - 18x - 10 = 0$$

Solving w/ calculator, $x \approx -1.753, -0.605, 2.358$.

From graph, $x > 0$, so $x \approx 2.358$, and $D \approx 2.701$ $\boxed{C}$

40. $x + y = \arctan(xy)$

$$1 + \frac{dy}{dx} = \frac{1}{1+(xy)^2} \cdot \left(x \frac{dy}{dx} + y \cdot 1\right)$$

$$1 + \frac{dy}{dx} = \frac{1x}{1+x^2y^2} \cdot \frac{dy}{dx} + \frac{y}{1+x^2y^2}$$ Multiply by $1+x^2y^2$

$$1 + x^2y^2 + \frac{dy}{dx}(1+x^2y^2) = x\frac{dy}{dx} + y$$

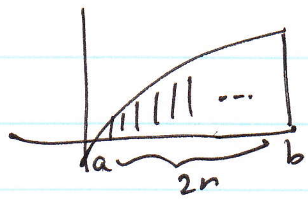
$$\frac{dy}{dx}(1+x^2y^2) - x\frac{dy}{dx} = y - 1 - x^2y^2$$

$$\frac{dy}{dx}(1+x^2y^2-x) = y-1-x^2y^2$$

$$\frac{dy}{dx} = \frac{y-1-x^2y^2}{1+x^2y^2-x}$$ $\boxed{E}$ ugh.

41. I find this question to be unreasonably difficult. If the sum went from $k=1$ to n, it would be fine; that's a standard Riemann sum. Changing the upper limit to $2n$ is nonstandard, and in my opinion ridiculous. I cannot imagine that this question would make it onto an AP Calculus exam, partly because they also disguised $\frac{1}{n}$ as $\sqrt{\frac{1}{n^2}}$, for no particular reason. The explanation in the back is fine, but here's another way to see it.

$\sum_{k=1}^{2n}$ means $2n$ pieces.



The width of each piece is then $\frac{b-a}{2n}$.

(continued)

41. (cont.) In general, a Riemann sum is $\sum_{k=1}^{n}(\text{base} \cdot \text{height})$. Since the width of

each piece is $\frac{b-a}{2n}$, that's the base. The heights are the function f evaluated at $a + k \cdot \left(\frac{b-a}{2n}\right)$, which gives right-hand heights. So now we have

$$\underbrace{\sum_{k=1}^{2n}\left[\left(\frac{b-a}{2n}\right) \cdot f\left(a+k\left(\frac{b-a}{2n}\right)\right)\right]}_{\text{my Riemann sum}} = \underbrace{\sum_{k=1}^{2n}\sqrt{\frac{1+\frac{k}{n}}{n^2}}}_{\text{given}} = \sum_{k=1}^{2n}\frac{\sqrt{1+\frac{k}{n}}}{n} = \sum_{k=1}^{2n}\frac{1}{n}\left(\sqrt{1+\frac{k}{n}}\right)$$

This now means that $\frac{1}{n} = \frac{b-a}{2n}$, so $b-a=2$.

I can eliminate everything except B and E, because those have limits of integration whose difference is 2. I also see that $f\left(a+k\left(\frac{b-a}{2n}\right)\right) = \sqrt{1+\frac{k}{n}}$, and

since both B and E have $a=0$ and $b=2$, this becomes

$$f\left(0+k\left(\frac{2}{2n}\right)\right) = f\left(\frac{k}{n}\right) = \sqrt{1+\frac{k}{n}}.$$

And therefore $f(x) = \sqrt{1+x}$, and the answer is $\boxed{B}$.

I don't like this question at all.

42. Total distance $= \int_a^b \text{speed}\, dt = \int_a^b |v(t)|\, dt$

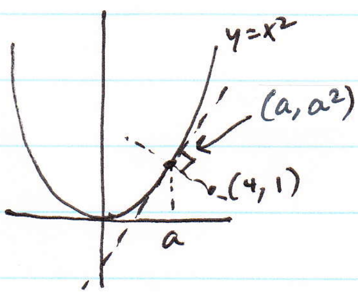
Particle A: $\int_0^4 |4-t^2|\, dt = 16$

$40-16 = 24$ $\boxed{A}$

Particle B: $\int_0^4 |t^3-4t|\, dt = 40$

43. BC only

44.



Slope of tangent at $x=a$ is from $y' = 2x$, so $y'(a) = 2a$.

Slope of line from (a, a^2) to $(4,1)$ is $\frac{1-a^2}{4-a}$. And those are perpendicular lines, so

$$\frac{1-a^2}{4-a} = \frac{-1}{2a}$$

$$2a - 2a^3 = -4 + a$$

$$-2a^3 + a + 4 = 0$$

Solving on a calculator, $a \approx 1.392$ $\boxed{A}$

45. The graph is $F(t) = \int_0^x f(t)dt$, so we see the integral values, not the values of f.

I. $\int_0^4 f(t)\,dt = F(4) = 0$, not 3. Nope.

II. $\int_2^4 f(t)\,dt = \int_0^4 f(t)\,dt - \int_0^2 f(t)\,dt$

$= F(4) - F(2) = 0 - \frac{3}{2} = -\frac{3}{2}$ Nope.

III. $\int_2^0 f(t)\,dt = -\int_0^2 f(t)dt = -F(2) = -\frac{3}{2}$

$\int_2^4 f(t)dt$ is $-\frac{3}{2}$ from part II. So yes. $\boxed{C}$

Free Response

1. a) Area of a trapezoid $= \frac{1}{2}h(b_1+b_2)$. Since times are measured in people per minute, the heights will be measured in minutes.

$$\left.\begin{aligned} A_{I} &= \tfrac{1}{2}\cdot 90\cdot(1+18) = 855 \\ A_{II} &= \tfrac{1}{2}\cdot 50\,(18+81) = 2475 \\ A_{III} &= \tfrac{1}{2}\cdot 30(81+60) = 2115 \\ A_{IV} &= \tfrac{1}{2}\cdot 10\,(60+44) = 520 \end{aligned}\right\}\ \text{total} = 5965 \text{ people}$$

b) Average value of number of people over that 50-minute period is $\frac{1}{50}\int_a^b \text{rate}\, dt$.

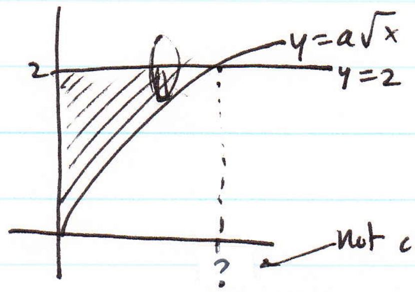
A right-hand Riemann sum is the area of 3 rectangles with heights taken at the ends of the intervals.

$$\frac{1}{50}\left(20\cdot 81 + 20\cdot 74 + 10\cdot 60\right) = \boxed{74 \text{ people/min}}$$

c) $\int_0^{180} R(t)\,dt \approx \boxed{5621 \text{ people}}$

d) $\frac{1}{50}\int_{120}^{170} R(t)\,dt \approx \boxed{73.895 \text{ people/min}}$

2. BC only

3. 

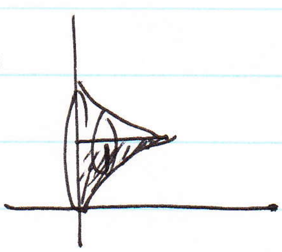
Not clear if that's at exactly 2.

a) Average value of f on $[0, 4] = \frac{1}{4}\int_0^4 a\sqrt{x}\,dx$

$$= \frac{1}{4}\cdot a\cdot \frac{2}{3}x^{3/2}\Big]_0^4 = \frac{1}{6}a\left(4^{3/2} - 0^{3/2}\right)$$

$$= \frac{1}{6}a\cdot 8 = \frac{4}{3}a$$

So $\frac{4}{3}a = 2$, and $a = 2\cdot\frac{3}{4} = \boxed{1.5}$

b) 

Discs: radius $= 2 - y = 2 - 1.5\sqrt{x}$

$$\pi\int_0^b r^2\,dx = \pi\int_0^b \left(2 - 1.5\sqrt{x}\right)^2 dx$$

But I still need the value of b, which is where

$$1.5\sqrt{x} = 2$$
$$\sqrt{x} = \frac{4}{3}$$
$$x = \frac{16}{9}.$$

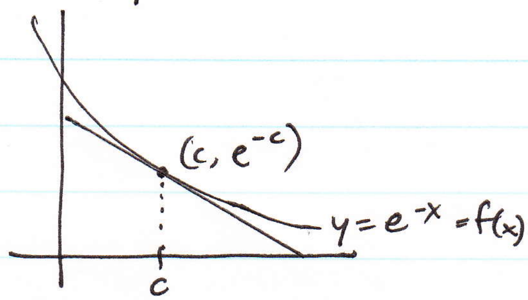
$\therefore \boxed{\pi\int_0^{16/9}\left(2 - 1.5\sqrt{x}\right)^2 dx}$

c) BC only

4. BC only

5. a) BC only

b)



$f'(x) = -e^{-x}$

Point (c, e^{-c})

Slope $f'(c) = -e^{-c}$

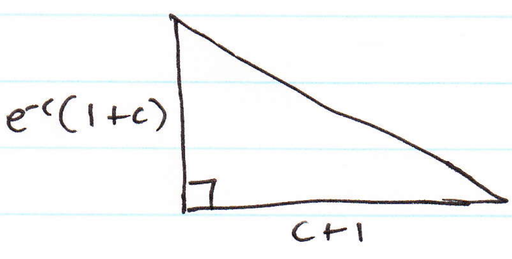
$$\boxed{\begin{array}{l}\text{Tangent line:}\\ y - e^{-c} = -e^{-c}(x-c)\end{array}}$$

y-intercept when $x=0$:

$$y - e^{-c} = -e^{-c}(0-c)$$
$$y - e^{-c} = +ce^{-c}$$
$$y = e^{-c} + ce^{-c} = e^{-c}(1+c)$$

x-intercept when $y=0$:

$$0 - e^{-c} = -e^{-c}(x-c)$$
$$-e^{-c} = -e^{-c}(x-c)$$
$$1 = x - c$$
$$x = c+1$$



$$\text{Area} = \frac{1}{2} \cdot (c+1) \cdot e^{-c}(1+c)$$
$$= \boxed{\frac{1}{2} e^{-c}(1+c)^2}$$

c) BC only

6. $xy + y^2 = x^2 - 5$

a) $x \cdot \frac{dy}{dx} + y \cdot 1 + 2y \cdot \frac{dy}{dx} = 2x$

$$x \frac{dy}{dx} + 2y \frac{dy}{dx} = 2x - y$$

$$(x + 2y) \frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx} = \frac{2x-y}{x+2y} \qquad \text{Q.E.D.}$$

b) Vertical tangents will happen when $\frac{dy}{dx} = \frac{\text{nonzero}}{\text{zero}}$.

$$x + 2y = 0$$

$$x = -2y$$

Substituting into the original equation gives

$$-2y \cdot y + y^2 = (-2y)^2 - 5$$

$$-2y^2 + y^2 = 4y^2 - 5$$

$$-y^2 = 4y^2 - 5$$

$$5 = 5y^2$$

$$y^2 = 1$$

$$y = \pm 1$$

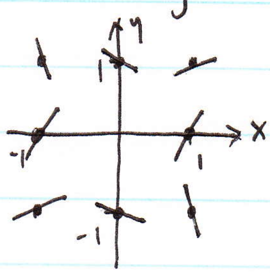
If $y = 1$, $x = -2 \cdot 1 = -2$

$$\boxed{(-2, 1)}$$

If $y = -1$, $x = -2 \cdot -1 = 2$

$$\boxed{(2, -1)}$$

c)



(x, y)	$\frac{dy}{dx}$
$(-1, 1)$	-3
$(0, 1)$	$-\frac{1}{2}$
$(1, 1)$	$\frac{1}{3}$

(x, y)	$\frac{dy}{dx}$
$(-1, 0)$	2
$(1, 0)$	2
$(-1, -1)$	$\frac{1}{3}$

(x, y)	$\frac{dy}{dx}$
$(0, -1)$	$-\frac{1}{2}$
$(1, -1)$	-3

d) $2x - y = 0 \Rightarrow \boxed{y = 2x}$