Practice Exam AB-3

A 1. $f(x) = e^{2\ln x} = e^{\ln x^2} = x^2$

$f'(x) = 2x$; $f'(3) = 6$

A 2. $f(x) = x - x^2$, $g(x) = x^3 - x$

Intersections at $(0,0)$, $(1,1)$ by thinking about it.

Or algebraically: $x - x^2 = x^3 - x$

$0 = x^3 + x^2 - 2x = x(x^2 + x - 2)$

$0 = x(x+2)(x-1)$

$x = 0, -2, 1$, but $x = -2$ outside interval.

The new #3 is the old #27. You will find it there.

$$\int_0^1 \left((x - x^2) - (x^3 - x)\right)dx = \int_0^1 (2x - x^2 - x^3)\,dx$$

$$= \left[x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1 = \left(1 - \tfrac{1}{3} - \tfrac{1}{4}\right) - 0 = \tfrac{12}{12} - \tfrac{4}{12} - \tfrac{3}{12} = \tfrac{5}{12}$$

B 4. $f'(x)$ = slope of graph.

$f'(1)$ = slope of $y = \sqrt{4 - x^2}$ at $x = 1$

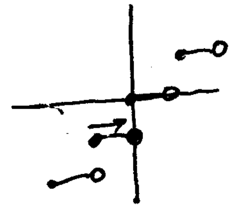
$$\tfrac{1}{2}(4 - x^2)^{-1/2} \cdot -2x \Big|_{x=1} = \tfrac{1}{2} \cdot 3^{-1/2} \cdot -2 = -\tfrac{1}{\sqrt{3}}$$

A 5. $f'(x) = 0$ where f has a horizontal tangent, at $x = 0$.

A 6. $s(t) = t^3 + 9t^2 - 27$ $\quad v(t) = 3t^2 + 18t$ $\quad a(t) = 6t + 18$

When $a(t) = 0$, $t = -3$. $\quad v(-3) = 3(-3)^2 + 18(-3) = +27 - 54 = -27$

B **7.** $\lim_{x\to 0^-} \frac{1}{[x]} = -1$



B **8.** $\frac{dy}{dx} = \sin(3x-3)+4$

$y = \int \sin(3x-3)\,dx + \int 4\,dx$

$y = -\frac{1}{3}\cos(3x-3) + 4x + C$ $\quad y(1)=7$

$7 = -\frac{1}{3}\cos 0 + 4\cdot 1 + C$

$7 = -\frac{1}{3} + 4 + C$

$3\frac{1}{3} = C \Rightarrow y = -\frac{1}{3}\cos(3x-3) + 4x + 3\frac{1}{3}$

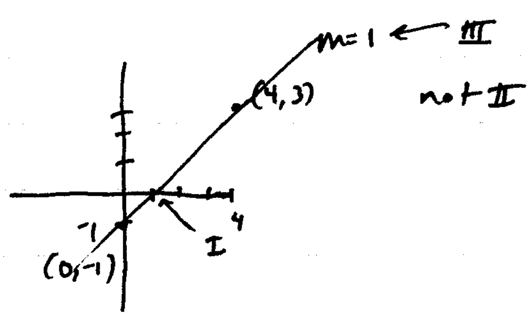
E **9.** $y = \tan^{-1}x$ at $x = \sqrt{3}$

$y' = \frac{1}{1+x^2}$ $\quad y'(\sqrt{3}) = \frac{1}{1+(\sqrt{3})^2} = \frac{1}{1+3} = \frac{1}{4}$

$y(\sqrt{3}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$y - \frac{\pi}{3} = \frac{1}{4}(x - \sqrt{3})$

D **10.** $f(0) = -1$ $\quad f(4) = 3$



B **11.** $f(x) = x^4 + ax^2$ has minimum at $x=2 \Rightarrow f'(2)=0$

$f'(x) = 4x^3 + 2ax$

$f'(2) = 4 \cdot 8 + 2 \cdot a \cdot 2 = 0$

$4a = -32 \Rightarrow a = -8$

A **12.** $-\int_{-\pi}^{0} \frac{-\sin x}{2+\cos x}\,dx$ $\quad u = 2+\cos x \quad x=-\pi \rightarrow u = 2+\cos(-\pi) = 1$

$du = -\sin x\,dx \quad x=0 \Rightarrow u = 2+\cos 0 = 3$

$= -\int_{1}^{3} \frac{1}{u}\,du = -\ln|u|\Big]_{1}^{3} = -\ln 3 - (-\ln 1) = -\ln 3$

C **13.** Zero when $y=0$ and $x=1 \Rightarrow$ C

D **14.** $y = e^{8x^2+1}$

$\frac{dy}{dx} = e^{8x^2+1} \cdot 16x$

D **15.** $y = xe^{-2x}$ when is $y'=0$?

$y' = xe^{-2x} \cdot -2 + e^{-2x} \cdot 1$

$= e^{-2x}(-2x+1) \Rightarrow -2x+1=0 \Rightarrow x = \frac{1}{2}$

$y(\frac{1}{2}) = \frac{1}{2} \cdot e^{-2 \cdot \frac{1}{2}} = \frac{1}{2}e^{-1}$

C **16.** Rectangle has area ~~2π~~ $2\pi \times 2 = 4\pi$

$$\int_0^{\pi} 2\sin x \, dx = -2\cos x\Big]_0^{\pi} = -2(\cos\pi - \cos 0) = -2(-1-1) = 4$$

$4\pi - 4$

C **17.** $\int (x^2+1)^2\,dx = \int (x^4 + 2x^2 + 1)\,dx = \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$

E **18.** This isn't really worded as an AB question.

What they're asking is which pair has a limit of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ that's either ∞ or 0, as opposed to a nonzero constant.

$$\lim_{x\to\infty} \frac{x^4}{x^3} = \lim_{x\to\infty} x = \infty$$

B **19.** $\int \sec^2(2x)\,dx \qquad u = 2x \qquad du = 2dx$

$$\frac{1}{2}\int 2\sec^2(2x)\,dx = \frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan u + C$$

$$= \frac{1}{2}\tan(2x) + C$$

D **20.** $f(x) = 3x^4 - 4x^3 + 6$ p.o.i. when f'' changes sign

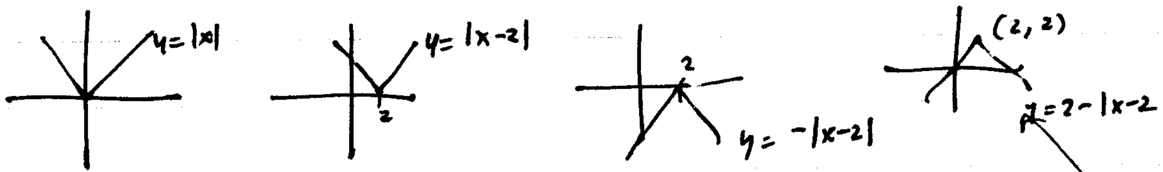
$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x = 0$

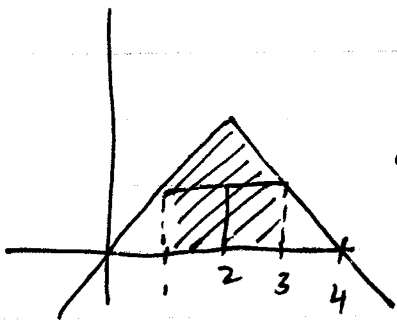
$12x(3x-2) = 0$

$x = 0, \ 2/3$ changes signs at both

D **21.** $\int_1^3 (2-|x-2|)\,dx$ Sneaky



$y = 2 - |x-2|$. If $x=0$, $y=0$. If $x=2$, $y=2$. If $x=4$, $y=0$.



area = 3

~~21 $y = |3x+1|$ $\frac{dy}{dx}$~~
~~$|x+1|$~~

A **22.** $y = \dfrac{3x+1}{x-1}$ $\quad y' = \dfrac{(x-1)3 - (3x+1)\cdot 1}{(x-1)^2} = \dfrac{\cancel{3x}-3-\cancel{3x}-1}{(x-1)^2} = \dfrac{-4}{(x-1)^2}$

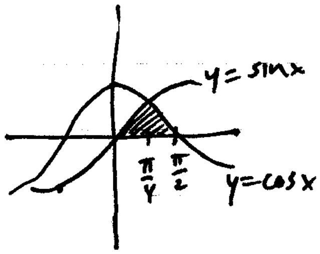
C **23.** $\dfrac{d}{dx}\int_x^2 \ln(1+t)\,dt = \dfrac{d}{dx}\left(-\int_2^x \ln(1+t)\,dt\right) = -\ln(1+x)$

E **24.** $f'(x) = -5(x-3)^2(x-2) = 0$ at $x=3, 2$

f': $+$ (2) $-$ (3) $-$ → maximum at $x=2$, NOT minimum at $x=3$, so E

These are a little out of order from the first edition, which is what I used when I wrote the solutions.

E **27.**



$$\int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx$$

E **28.** $\int_0^6 f(x)dx =$ 3.5 above axis + −2 below = 1.5

C 26 $\int_1^k \frac{1}{\sqrt{x}} dx = 4$

$$\int_1^k x^{-1/2} dx = 2x^{1/2}\Big]_1^k = 2(\sqrt{k} - \sqrt{1}) = 4$$

$$\sqrt{k} - 1 = 2$$

$$\sqrt{k} = 3 \Rightarrow k = 9$$

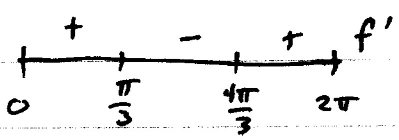
B **3.** $f(x) = e^{x\sqrt{3}} \cos x$ f is decreasing when $f' < 0$

$$f'(x) = e^{x\sqrt{3}} \cdot -\sin x + \cos x \cdot e^{x\sqrt{3}} \cdot \sqrt{3} = 0$$

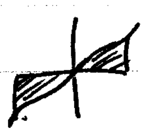
$$e^{x\sqrt{3}}(-\sin x + \sqrt{3}\cos x) = 0$$

$$\sqrt{3}\cos x = \sin x$$

$\sqrt{3} = \tan x \Rightarrow x = \frac{\pi}{3}, \frac{4\pi}{3}$ on given interval,



A **25.** Origin symmetry:



cancel at $\int_{-b}^{b} f(x)dx = 0$

E 29. $x + \sin y = \ln y$

$1 + \cos y \cdot \frac{dy}{dx} = \frac{1}{y} \cdot \frac{dy}{dx}$

$1 = \left(\frac{1}{y} - \cos y\right)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\frac{1}{y} - \cos y} \cdot \frac{y}{y} = \frac{y}{1 - y\cos y}$

B 30. $y = \sin(x - \sin x)$ tangent $\parallel$ to x-axis $\Rightarrow$ $y' = 0$

$y' = \cos(x - \sin x) \cdot (1 - \cos x) = 0$

on calculator, $x \approx 2.30988$

C 31. $f(g(x)) = x \Rightarrow f, g$ are inverses.

$f(a) = b \Rightarrow g(b) = a$. So I can conclude that

$g'(b) = \frac{1}{f'(a)} = \frac{1}{c}$

E 32. $h(t) = -\cos\left(\frac{\pi t}{12}\right) + 4$

If $t = 0$ is midnight, then noon is $t = 12$ and 3 p.m. is $t = 15$.

$\frac{1}{15-12}\int_{12}^{15} h(t)\,dt \approx 4.900$

A 33. ~~[illegible]~~

$$\begin{array}{ccc} & - & + \quad f' \\ \hline -7 & -5 & -4 \end{array}$$

~~maximum~~ minimum!

D 34. A: no,

B: no,

C: no, see above

D: This one! If $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ then $\lim_{x\to 3} f(x)$ does exist.

E: no

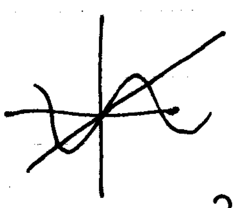
D 35. $v(t) = 5 - e^t$

total distance $= \int_0^3 |v(t)|\,dt \approx 12.1799$

C 36. $f' < 0 \Rightarrow f$ decreasing

$f'' < 0 \Rightarrow f$ cc down

D 37.



Symmetric, so find one side and double it. $x = \sin 3x \Rightarrow x \approx 0.7596\ldots = g$

$2 \cdot \int_0^g (\sin(3x) - x)\,dx \approx 0.523$

C 38. $f(x+y) = f(x)\cdot f(y)$ $\qquad \lim_{h\to 0} \frac{f(h)-1}{h} = 6.$

$f'(x) = ?$ This is harder than would be on the AB exam, I think. Def'n of derivative:

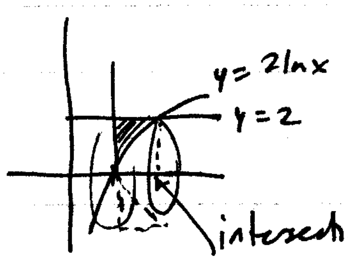
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\underline{f(x)f(h)} - f(x)}{h}$$ (using that rule about $f(x+y)$)

$$= \lim_{h\to 0} \frac{f(x)\left(f(h)-1\right)}{h}$$

$$= f(x)\cdot 6$$

E 39.



$$\pi \int_1^e (R^2 - r^2)\,dx$$
$$= \pi \int_1^e \left(2^2 - (2\ln x)^2\right)dx$$

intersection at $2 = 2\ln x$
$\ln x = 1$
$x = e$

$R = 2$
$r = 2\ln x$

B 40. $\int_2^5 f(x)\,dx \approx \frac{1}{2}\cdot 1\cdot(0.21+0.28) + \frac{1}{2}(1)(0.28+0.36) + \frac{1}{2}\cdot 1\,(0.36+0.44)$
$= 0.965$

C 41. $f' > 0$, but nearly zero at about $x = 3$
f increases.

D 42. $\frac{dr}{dt} = 2$ cm/min $\quad V = 40$ ~~cm~~ cm^3, find $\frac{dV}{dt}$.

If $V = 40 = \frac{4}{3}\pi r^3$, then $r = \sqrt[3]{\frac{40}{\pi}\cdot\frac{3}{4}}$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

At that time $\frac{dV}{dt} = 4\pi\left(\sqrt[3]{\frac{40}{\pi}\cdot\frac{3}{4}}\right)^2 \cdot 2 \approx 113.124$

C 43. $\int_3^7 g(x)\,dx = 6 \qquad \int_3^7 h(x)\,dx = 2 \qquad \int_3^4 h(x)\,dx = -4$

I: $\int_4^7 h(x)\,dx = \int_3^7 h(x)\,dx - \int_3^4 h(x)\,dx = 2-(-4) = 6.$ True

II: $\int_3^7 g(x)\,h(x)\,dx \longrightarrow$ there's no useful rule for this.

Doesn't have to be true.

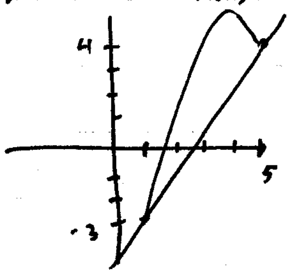
III: $\int_3^7 (g(x)+2)\,dx = \int_3^7 g(x)\,dx + \int_3^7 2\,dx$

$$= 6 + 2\cdot 4 = 14.$$ True

D 44. $\frac{dy}{dx} = \frac{\sin x}{x} \qquad y(1) = 4$

$$y(2) = y(1) + \int_1^2 \frac{\sin x}{x}\,dx \approx 4.659$$

E 45. $f(1) = -3$ $f(5) = 4$



A: $f(0)$? I know nothing outside of $[1, 5]$ about y-values, so no.

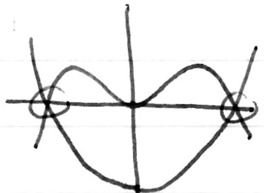
B: $f(x)$ increasing, not necessarily

C: $f'(x)$ doesn't have to be constant

D: $f'(k) = 0$, no

E: $f'(k) = \frac{7}{4}$ someplace? $\frac{4-(-3)}{5-1} = \frac{7}{4}$ yes, MVT

FR 1. a) S:



Does this look like a tongue to you?

x-intercepts are the limits of integration.

$\sin(x^2) = 0 \Rightarrow x \approx \pm 1.77245, 0$

$$\int_{-1.77245}^{1.77245} \left(\sin(x^2) - g(x)\right)dx = \int_{-1.77245}^{1.77245} \sin(x^2)\,dx + 2R$$

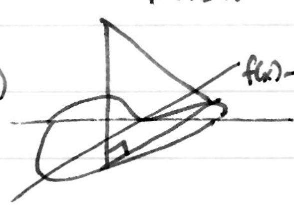
$$\approx 1.78966 + 2(3.712) \approx 9.21366 \approx 9.214$$

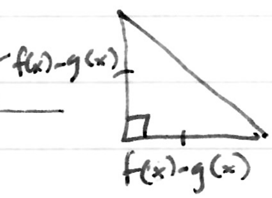
b) outer radius $= 2 - g(x)$ (because $g(x) < 0$, $2 - g(x) > 0$)

inner radius $= 2 - f(x)$

$$\pi \int_{-1.77245}^{1.77245} \left((2-g(x))^2 - (2-f(x))^2\right)dx$$

c)





Area: $\frac{1}{2}(f(x) - g(x))^2$

Volume: $\displaystyle\int_{-1.77245}^{1.77245} \frac{1}{2}(f(x) - g(x))^2\,dx$

top $= f(x)$, bottom $= g(x)$; distance between $=$ top $-$ bottom

FR2. a) $\int_0^{45} N(t)\,dt$ will approximate the total number of cases of the illness from $t=0$ to $t=45$.

$2\cdot 3 + 4\cdot 8 + 4\cdot 15 + 5\cdot 30 + 10\cdot 100 + 10\cdot 50 + 5\cdot 22 + 5\cdot 10$

$= \boxed{1908 \text{ cases}}$

b) Average number of cases (preamably on a given day? not entirely clear). Assuming that,

$$\frac{1}{45}\int_0^{45} R(t)\,dt \approx \cancel{43.14}\ \boxed{43.153 \text{ cases per day}}$$

This does match BoB, but the question is quite poorly worded. Should have said "average number of ~~cas~~ cases per day."

c) Solve $R(t) = 5$ on calculator, where $t > 45$.

$\boxed{t \approx 48.274}$, or approximately day 48

(Not clear if supposed to be to the nearest day)

FR 3. $g(x) = \int_0^{2x} f(t)\,dt$ Sneaky, 2x.

a) $g(2) = \int_0^4 f(t)\,dt = \frac{1}{2} + -\frac{1}{2}\pi + \frac{1}{2} = \boxed{1 - \frac{1}{2}\pi}$

$g'(\cancel{x}) = f(2x)\cdot 2$

$g'(2) = f(4)\cdot 2 = 1\cdot 2 = \boxed{2}$

b) g has local extrema when g' changes signs

$f(2x)\cdot 2 = 0 \Rightarrow f(2x) = 0$

so $2x = 1$, $2x = 3$

$x = \frac{1}{2}$ $x = \frac{3}{2}$

At $\boxed{x = \frac{1}{2}}$, g' changes from + to −, so $\boxed{\text{maximum}}$

At $\boxed{x = \frac{3}{2}}$, g' changes from − to +, so $\boxed{\text{minimum}}$

c) g(x) is concave up when $g'' > 0$.

$g'(x) = 2f(2x)$, so

$g''(x) = 4f'(2x)$.

~~g'' > 0 when f slopes up, on (2, 1) and on (3, 4)~~,

$g''(x) > 0$ when $f(2x) > 0$

So $-2 < 2x < -1 \Rightarrow$ $\boxed{-1 < x < -\frac{1}{2}}$

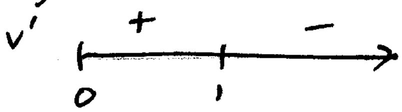
and $3 < 2x < 4 \Rightarrow$ $\boxed{\frac{3}{2} < x < 2}$

FR4. $v(t) = \dfrac{3t}{1+t^2}$, $t \geq 0$ $\qquad x(0) = 4.$

a) Maximum velocity will occur at an endpoint or when v' changes $+$ to $-$.

$$v'(t) = \frac{(1+t^2)3 - 3t \cdot 2t}{(1+t^2)^2} = \frac{3+3t^2-6t^2}{(1+t^2)^2} = \frac{3-3t^2}{(1+t^2)^2}$$

$v'(t)=0 \Rightarrow 3-3t^2=0 \Rightarrow t=\pm 1$, but only $t=1$ on given domain.



Since $v'>0 \ \forall\, t<1$ and $v'<0 \ \forall\, t>1$, the maximum occurs at $t=1$.

$$v(1) = \frac{3 \cdot 1}{1+1^2} = \boxed{\frac{3}{2}}$$

b) $x(t) = \displaystyle\int \frac{3t}{1+t^2}\,dt = \frac{3}{2}\int \frac{2t}{1+t^2}\,dt = \frac{3}{2}\int \frac{1}{u}\,du$

$u = 1+t^2$ $\qquad = \frac{3}{2}\ln|1+t^2| + C$

$du = 2t\,dt$

Since $x(0)=4$, $\quad 4 = \frac{3}{2}\ln|1| + C \Rightarrow C=4$,

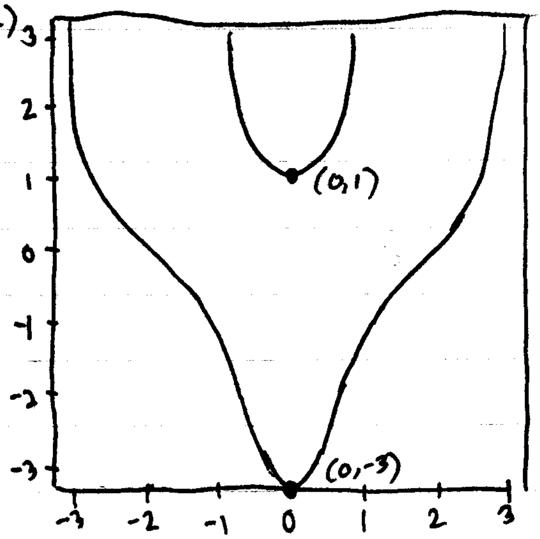
and $\boxed{x(t) = \frac{3}{2}\ln(1+t^2)+4}$

c) $\displaystyle\lim_{t\to\infty} v(t) = \boxed{0}$

d) $\displaystyle\lim_{t\to\infty} x(t) = \boxed{\infty}$

FR 5. $\frac{dy}{dx} = x(y^2+1)$

a)



b) $\frac{d^2y}{dx^2} = x \cdot 2y \cdot \frac{dy}{dx} + (y^2+1) \cdot 1$

$= 2xy \cdot x(y^2+1) + (y^2+1)$

$$\boxed{\frac{d^2y}{dx^2} = (y^2+1)(2x^2y+1)}$$

At a point of inflection, $\frac{d^2y}{dx^2}$ changes signs.

Since $y^2+1 > 0$, $2x^2y+1$ must change signs

$2x^2y + 1 = 0$

$2x^2y = -1$

$y = \frac{-1}{2x^2}$. Since $x^2 \geq 0$, $y < 0$ at any potential inflection point.

c) $\int \frac{dy}{y^2+1} = \int x\,dx$

$\arctan y = \frac{1}{2}x^2 + C$

$y = \tan\left(\frac{1}{2}x^2 + C\right)$

At $(0,1)$, $1 = \tan\left(\frac{1}{2}\cdot 0^2 + C\right)$

$\tan C = 1 \Rightarrow c = \frac{\pi}{4}$

$$\Rightarrow \boxed{y = \tan\left(\frac{1}{2}x^2 + \frac{\pi}{4}\right)}$$

FR6. I screwed this up the first time, so I'm starting again. (couldn't read my own handwriting)

Ellipse: $\frac{x^2}{16}+\frac{y^2}{9}=1$ Hyperbola: $\frac{x^2}{9}-\frac{y^2}{16}=1$

Clearing fractions,

Ellipse : $9x^2+16y^2=144$ Hyperbola: $16x^2-9y^2=144$

a) Ellipse: $18x+32y\frac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{-18x}{32y}=\boxed{\frac{-9x}{16y}}$$

b) Slope: $\frac{-9\cdot\frac{12}{5}}{16\cdot\frac{12}{5}}=\frac{-9}{16}$ $\boxed{y-\frac{12}{5}=\frac{-9}{16}\left(x-\frac{12}{5}\right)}$

c) Hyperbola: $32x-18y\frac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{-32x}{-18y}=\boxed{\frac{16x}{9y}}$$

d) Slope $=\frac{16}{9}=\frac{16x}{9y}\Rightarrow x=y$

Substituting into hyperbola, $16x^2-9x^2=144$

$$7x^2=144$$

$$x=\pm\sqrt{\frac{144}{7}}=\pm\frac{12}{\sqrt7}$$

and $x=y$.

$\boxed{\left(\frac{12}{\sqrt7},\frac{12}{\sqrt7}\right)}$ is a point described