Practice Exam AB-2

E 1. $f(x) = x^2 \ln x$

$f'(x) = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x = x + 2x \ln x$

B 2. $\int_3^{11} \frac{1}{\sqrt{2x+3}}\,dx \qquad u = 2x+3 \qquad du = 2dx$

$x = 3 \Rightarrow u = 9, \quad x = 11 \Rightarrow u = 25$

$\frac{1}{2}\int_9^{25} u^{-1/2}\,du = \cancel{\frac{1}{2}\,u} \ \frac{1}{2} \cdot 2u^{1/2}\Big]_9^{25} = \sqrt{25} - \sqrt{9} = 2$

B 3. $\lim_{x \to c} f(x) = f(c) \Rightarrow f$ is continuous on $[0,5]$.

But that says nothing about differentability...

B 4. $\lim_{x \to 0^+} \frac{12}{4+e^{\csc x}} = \frac{12}{4+e^{\csc 0}} = \frac{12}{4+e^{\text{undef}}} = \cancel{\frac{12}{4+\infty}} = 0$ ~~does not exist~~

$\csc 0 = \frac{1}{\sin 0} = \frac{1}{0}$ is undefined, but $\lim_{x \to 0^+} \csc x = \infty$

A 5. Slope is 0 when $y = 0$, and never undefined.

Slope is 0 when $x = 0$, too.

C 6. $\frac{d}{dx}\left[2^{\cos x}\right] = 2^{\cos x} \cdot \ln 2 \cdot -\sin x$

E 7. $\int \frac{1}{\sqrt{e^x}}\,dx = \int (e^x)^{-1/2}\,dx = \int e^{-\frac{1}{2}x}\,dx = -2e^{-1/2\,x} + C$

$= \frac{-2}{\sqrt{e^x}} + C$

D 8. $f'(x) = x^3(x+2)^2$ $\quad$ f has inflection pts...

$f''(x) = x^3 \cdot 2(x+2)' + (x+2)^2 \cdot 3x^2$

$= (x+2) \cdot x^2 \left(2x + 3(x+2)\right) = (x+2) \cdot x^2 \cdot (5x+6)$

$f'' = 0$ at $x = -2, 0, -\frac{6}{5}$

f'' changes signs at $x = -2, \frac{-6}{5}$

D 9. $f(x) = \int_2^{\sin x} \sqrt{1+t^2}\, dt$

$f'(x) = \sqrt{1+(\sin x)^2} \cdot \cos x$

B 10. From diagram, $\int_a^0 f(x)dx = 5.$ $\quad \int_a^b f(x)dx = 3.$

$\int_a^b f(x)dx = \int_a^0 f(x)\, dx + \int_0^b f(x)dx$

$3 = 5 + \int_0^b f(x)\, dx \Rightarrow \int_0^b f(x)dx = -2$

A 11. $\lim\limits_{x\to -\infty} \frac{2x+3}{\sqrt{x^2+x+1}}$ $\quad \frac{2x}{|x|} = -2$

A **13.** $\frac{dy}{dx} = \frac{x}{\cos y} \Rightarrow \int \cos y\, dy = \int x\, dx$

The new #12 is the old #24. You will find it after #23.

$\sin y = \frac{1}{2}x^2 + C$

$y = \arcsin\left(\frac{1}{2}x^2 + C\right)$

$(1,0) \rightarrow \sin 0 = \frac{1}{2} \cdot 1^2 + C$

$0 - \frac{1}{2} = C = -\frac{1}{2}$, so $y = \arcsin\left(\frac{1}{2}x^2 - \frac{1}{2}\right)$

D **14.** $\frac{d}{dx}\left(\sin^{-1}(\ln x)\right) = \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{1}{x}$

C **15.** $g(x) = \frac{3x^2}{e^{3x}}$ $\quad$ g is increasing when...

$$g'(x) = \frac{e^{3x}\cdot 6x - 3x^2 \cdot 3e^{3x}}{e^{6x}} = \frac{e^{3x}(6x-9x^2)}{e^{6x}}$$

$$= \frac{3x(2-3x)}{e^{3x}}. \quad \text{Since } e^{3x} > 0 \ \forall x,$$

Need $3x(2-3x) > 0$ $\qquad$ $-\ \ +\ \ -\ \ g'$; 0, $2/3$

$x = 0, \quad x = \frac{2}{3}$

$x \in \left[0, \frac{2}{3}\right]$

E **16.** $x^2y + y^2 + 4 = 0 \qquad x = 2 \qquad 4y + y^2 + 4 = 0$

$x^2 \cdot \frac{dy}{dx} + y \cdot 2x + 2y \cdot \frac{dy}{dx} = 0 \qquad y^2 + 4y + 4 = 0$

$y = -2$

$$(x^2 + 2y)\frac{dy}{dx} = -2xy$$

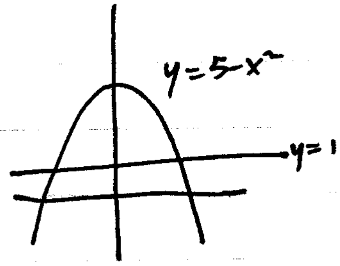
$$\frac{dy}{dx} = \left.\frac{-2xy}{x^2+2y}\right|_{(2,-2)} = \frac{-2\cdot 2\cdot -2}{(2)^2 + 2(-2)} = \frac{\quad}{0} \quad \text{undefined}$$

E **17.** $y = \tan 2x, \qquad x = \frac{\pi}{8}$

$y' = 2\sec^2 2x$

$y'\left(\frac{\pi}{8}\right) = 2\sec^2\frac{\pi}{4} = 2 \cdot 2 = 4$

E 18.



$5-x^2=1$

$4=x^2$

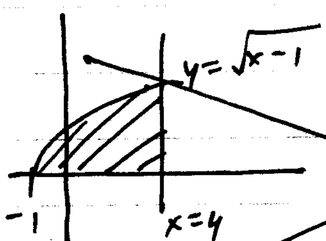
$x=\pm 2$

$$\int_{-2}^{2}(5-x^2-1)\,dx = \int_{-2}^{2}(4-x^2)\,dx = \left[4x-\tfrac{1}{3}x^3\right]_{-2}^{2}$$

$$= \left(8-\tfrac{8}{3}\right)-\left(-8+\tfrac{8}{3}\right) = 16-\tfrac{16}{3} = \tfrac{32}{3}$$

A 19. $\frac{1}{4-0}\int_0^4 e^{3x}\,dx = \frac{1}{4}\cdot\frac{1}{3}e^{3x}\Big]_0^4 = \frac{1}{12}\left(e^{12}-e^{0}\right)$

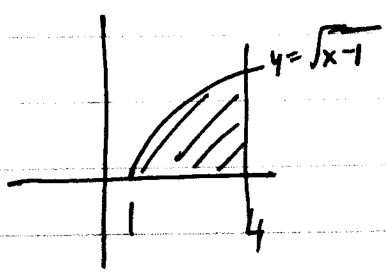
D 20.



$$\pi\int_{-1}^{4}\left(\sqrt{x-1}\right)^2 dx = \pi\int_{-1}^{4}(x-1)\,dx$$

$$= \pi\left(\tfrac{1}{2}x^2-x\right)\Big]_{-1}^{4}$$

$$= \pi\left[(8-4)-\left(\tfrac{1}{2}+1\right)\right] = \pi\left[4-1.5\right] = 2.5\pi$$



$$\pi\int_{1}^{4}\left(\sqrt{x-1}\right)^2 dx = \pi\int_{1}^{4}(x-1)\,dx$$

$$= \pi\left(\tfrac{1}{2}x^2-x\right)\Big]_{1}^{4}$$

$$= \pi\left[(8-4)-\left(\tfrac{1}{2}-1\right)\right] = \pi\left(4+\tfrac{1}{2}\right)$$

D 21. $x^2-y^2=5$ Find $\frac{d^2y}{dx^2}$ at $(3,2)$.

$2x - 2y\cdot\frac{dy}{dx}=0$

$\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$

$\frac{d^2y}{dx^2} = \frac{y\cdot 1 - x\cdot\frac{dy}{dx}}{y^2}$

At $(3,2)$, $\frac{dy}{dx} = \frac{3}{2}$

At $(3,2)$, $\frac{d^2y}{dx^2} = \frac{2\cdot 1 - \frac{3}{2}\cdot 3}{2^2} = \frac{2-\frac{9}{2}}{4} = \frac{-5/2}{4} = \frac{-5}{8}$

C 22. $v(t)=t^3-\sin t+2$. Find $a(2\pi)$.

$v'(t)=a(t)=3t^2-\cos t$

$a(2\pi) = 3(2\pi)^2 - \cos 2\pi = 12\pi^2-1$

C 23. $\frac{1}{2}\int_0^{\sqrt{e-1}} \frac{2x}{x^2+1}\,dx$ $u=x^2+1$ $du=2x\,dx$

$x=0 \Rightarrow u=1;$ $x=\sqrt{e-1} \Rightarrow u=e-1+1=e$

$\frac{1}{2}\int_1^e \frac{du}{u} = \frac{1}{2}\ln|u|\Big]_1^e = \frac{1}{2}(\ln e - \ln 1) = \frac{1}{2}(1-0) = \frac{1}{2}$.

C 12. $f(\frac{\pi}{4}) = f(\frac{\pi}{3}) + \int_{\pi/3}^{\pi/4} \tan x\,dx = \ln 6 + \ln|\sec x|\Big]_{\pi/3}^{\pi/4}$

$= \ln 6 + \ln(\sec\frac{\pi}{4}) - \ln(\sec\frac{\pi}{3})$

$= \ln 6 + \ln(\sqrt{2}) - \ln(2) = \ln(\frac{6\sqrt{2}}{2}) = \ln(3\sqrt{2})$

B 24. $y = x^2$ $\qquad$ $y = \sqrt{x}$

$y' = 2x$ $\qquad$ $y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

When are these equal?

$$2x = \frac{1}{2\sqrt{x}}$$

$$(4x\sqrt{x})^2 = (1)^2$$

$$16x^2 \cdot x = 1$$

$$16x^3 = 1$$

$$x = \sqrt[3]{\frac{1}{16}} \cdot \sqrt[3]{\frac{4}{4}} = \frac{\sqrt[3]{4}}{\sqrt[3]{64}} = \frac{\sqrt[3]{4}}{4}$$

E 25. $f(x) = \frac{\ln x}{x}$ $\qquad$ Decreasing when $f'(x) < 0$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

f' undefined when $x \leq 0$.

$f' = 0 \Rightarrow 1 - \ln x = 0$ $\qquad$ $+$ $\quad$ $-$ $\quad$ f'

$\ln x = 1$ $\qquad$ 0 $\quad$ e

$x = e$ $\qquad$ Decreasing on $[e, \infty)$

E 26. $\lim\limits_{x \to 2} \frac{e^{2x} - e^4}{x - 2}$ $\qquad$ $\frac{0}{0}$, L'Hôpital's

$$= \lim_{x \to 2} \frac{2e^{2x}}{1} = 2e^4$$

E 27. $\displaystyle\int_0^{\pi^2/9} \frac{dx}{\sqrt{x}\,\cos^2(\sqrt{x})}$ $\qquad u = \sqrt{x}\,;\quad du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}\,dx$

$x = 0 \Rightarrow u = 0$

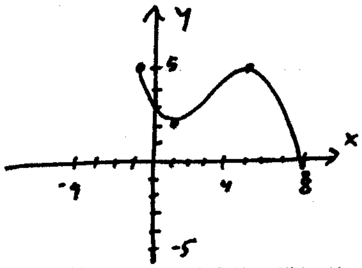
$x = \frac{\pi^2}{9} \Rightarrow u = \frac{\pi}{3}$

$$= 2\int_0^{\pi^2/9} \frac{dx}{2\sqrt{x}\cos^2\sqrt{x}}$$

$$= 2\int_0^{\pi/3} \sec^2 u \, du = 2\tan u\Big]_0^{\pi/3}$$

$$= 2\left(\tan\tfrac{\pi}{3} - \tan 0\right) = 2(\sqrt{3} - 0) = 2\sqrt{3}$$

? A 28.

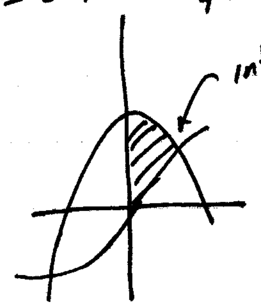


Only one zero required $\Rightarrow$ not D

Only one p.o.i. required $\Rightarrow$ not C, E

Depends on definition of relative maximum. Could be A, if endpt. not included. Otherwise B.

B 29. $y = 2 - x^2 \qquad y = 3\sin x \qquad$ y-axis



intersection $\quad 2 - x^2 = 3\sin x$

$x \approx 0.585\ldots$

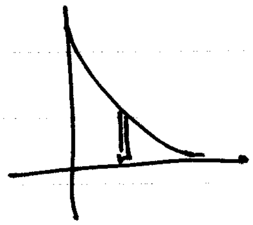
$$\int_0^{0.585\ldots} (2 - x^2 - 3\sin x)\,dx \approx 0.604$$

A 30. $h(x) = \dfrac{f(x)}{g(x)}$. In general, $h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

But here, $g(x)f'(x) - f(x)g'(x) = f(x)\left(g(x) - g'(x)\right)$.

$\therefore f(x) = f'(x) \Rightarrow A$

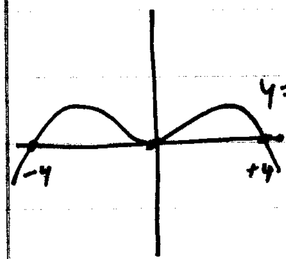
C 31. $y=(3-x)e^{-x}$



$$\int_0^3 (\text{side})^2\,dx = \int_0^3 \left((3-x)e^{-x}\right)^2 dx$$

$$\approx 3.249$$

B 32.



Horz. tangents at $x=0,\ 4,\ -4$.

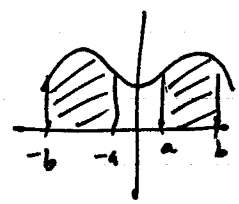
So not A, D, E.

Positive on $(-4,\ 4)$, so increasing. $\Rightarrow$ B

E 33. $f(x)$ is even $\Rightarrow$ $f(-x)=f(x)$. $\Rightarrow$ I true

However, $f(x)$ might still be negative, so not II.

$$\int_{-b}^{-a} f(x)\,dx = \int_a^b f(x)\,dx \Rightarrow \text{III true}$$



B 34. A) No, hole c) No, different L/R e) No, hole

B) True d) " " "

A 35. $y=Ce^{kt}$ $(2, 500)$ $(6, 1500)$.

$(0, 500)$ $(4, 1500)$ translated, b/c doesn't matter

$y=500e^{kt}$

$3=e^{k\cdot 4}$

$k=\frac{1}{4}\ln 3$

$2=e^{(\frac{1}{4}\ln 3)t}$

$\ln 2=\left(\frac{1}{4}\ln 3\right)t$

$t\approx 2.524$

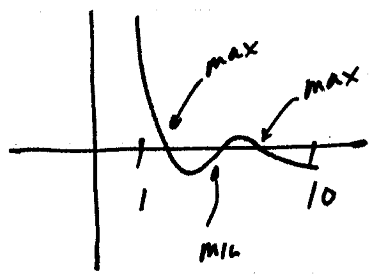
(Duh. If triples in 4 h, doubles in less than 4 h)

C 36. $g(x) = \int_{-4}^{x} f(t)\,dt$

$g(4) = \int_{-4}^{4} f(t)\,dt = \frac{1}{2}\cdot 6\cdot 6 + -\frac{1}{2}\cdot \pi \cdot 1^2$

$= 18 - \frac{1}{2}\pi$

C 37. $f'(x) = \frac{8\cos x}{x^2} - \frac{1}{8}$



D 38. $\frac{1}{2}\cdot 3(16+10) + \frac{1}{2}\cdot 5(10+6) + \frac{1}{2}\cdot 2(6+5)$

$= 13\cdot 3 + 8\cdot 5 + 11 = 39 + 40 + 11 = 90$

B 39. $x(t) = \frac{t^3}{3} + 2t + 2 \qquad t \in [0,3]$

$v(t) = t^2 + 2$

Average velocity on $[0,3]$ $= \frac{1}{3}\cdot(x(3) - x(0)) = 5$

$t^2 + 2 = 5$

$t^2 = 3 \Rightarrow t = \sqrt{3}$

D 40. $f'(x) = \sqrt{1+x^3}$ $\quad f(1) = 0.5$ $\quad$ Find $f(4)$.

$$f(4) = f(1) + \int_1^4 f'(x)\,dx \approx 13.371$$

C 41. f is decreasing $\Rightarrow$ $f' < 0$

f is cc up $\Rightarrow$ $f'' > 0$

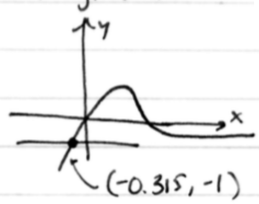
f has x-int $\Rightarrow$ $f = 0$. $\quad \therefore\ f' < f < f''$

E 42. $f(x) = x^2 e^{-x} - 2$

$f'(x) = -1$

Graphing $\frac{d}{dx}\left(x^2 e^{-x} - 2\right)$ and $y = -1$ gives an intersection at $x \approx -0.315$

(The answer in the back is wrong.)



A 43. $\cancel{\frac{\pi}{n}\sum_{i=1}^{n}\left(\right.}$ $\quad \frac{\pi}{n}\sum_{i=1}^{n}\sin\left(\frac{\pi i}{n}\right)$

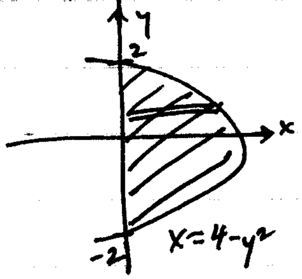
when $i = 1$, gives $\frac{\pi}{n} \cdot \sin\left(\frac{\pi}{n}\right)$.

When $i = n$, gives $\frac{\pi}{n} \cdot \sin(\pi)$

There are n subintervals, so likely interval is π units wide, $\sin x$ from $[0, \pi]$. I hate these.

D 44. $x = 4 - y^2$

Revolved about y-axis.



$$\pi \int_{-2}^{2} (4-y^2)^2 \, dy \approx 107.233$$

D 45. $V = \frac{4}{3}\pi r^3$ $\qquad S = 4\pi r^2$

$\frac{dr}{dt} = 5$ $\qquad$ Find $\frac{ds}{dt}$ when $V = 36\pi$.

$V = 36\pi = \frac{4}{3}\pi r^3 \Rightarrow r = 3.$

$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = 8\pi \cdot 3 \cdot 5 = 120\pi$

FR

FR 1. a) $\int_0^{1.5}\left(2e^{-\frac{x^2}{3}} - g(x)\right)dx = \int_0^{1.5} 2e^{-\frac{x^2}{3}}dx - \int_0^{1.5} g(x)dx$

$\approx 2.39252 - \frac{3}{5} \approx 1.793$

b) 

$R = 1 + f(x)$

$r = 1 + g(x)$

$\pi\int_0^{1.5}\left((1+f(x))^2 - (1+g(x))^2\right)dx$

c) 

radius of semicircle $= \frac{1}{2}g(x)$

$\frac{1}{2}\cdot\pi\int_0^{1.5}\left(\frac{1}{2}g(x)\right)^2dx = \pi\cdot\frac{1}{8}\int_0^{1.5}(g(x))^2dx$

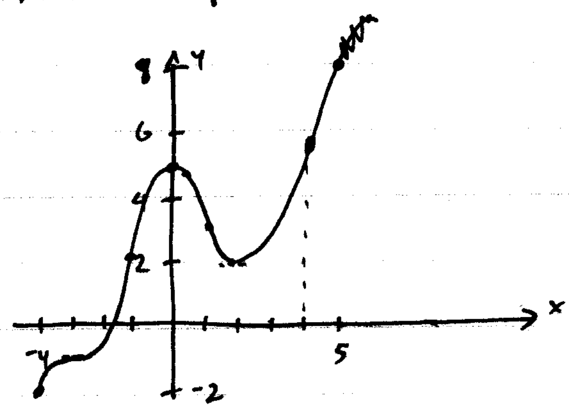
b/c semicircle

$= \frac{\pi}{8}\cdot\frac{3}{11} = \frac{3\pi}{88}$

FR 3. a) f is decreasing when $f' \leq 0$. $\Rightarrow \boxed{[0,2]}$

b) f has relative maximum when f' changes + to − $\Rightarrow \boxed{x=0}$

c) f is cc up when f' is increasing $\Rightarrow \boxed{(-3,-1) \text{ and } (1,4)}$

d) 

2. a) Average decay rate $= \frac{1}{24-0}\int_0^{24} R(t)\,dt$

$$\approx \frac{1}{24}\left[8 \cdot 221 + 8 \cdot 82 + 8 \cdot 22\right] = \frac{325}{3} \approx \boxed{108.333 \text{ g/min}}$$

b) $R'(12) \approx \frac{R(16)-R(8)}{16-8} = \frac{39-130}{16-8} = \boxed{-11.375 \text{ g/min}^2}$

c) $G(t) = 320(0.882)^t$

$$\frac{1}{24-0}\int_0^{24} G(t)\,dt = \boxed{100.972 \text{ grams per minute}}$$

d) $G'(12) \approx \boxed{-8.905 \text{ g/min}^2}$

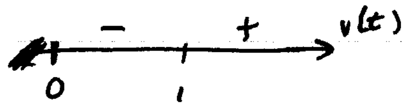
The rate of decay is decreasing at -8.905 grams per minute each minute when $t=12$ min.

4. $y = t^2 - 4\ln(t+1) - 1$

a) $v(t) = y'(t) = 2t - 4 \cdot \frac{1}{t+1} = \boxed{2t - \frac{4}{t+1}}$

b) Smallest value of y occurs either at an endpoint or when v changes signs from $-$ to $+$.

$v(t) = 0 = 2t - \frac{4}{t+1}$



$2t = \frac{4}{t+1}$

$2t^2 + 2t - 4 = 0$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

$\cancel{t = -2}$, $t = 1$

Since position decreases $(v<0)$ beginning at $t=0$, and v only changes sign $-$ to $+$ at $t=1$, the minimum velocity occurs at $t=1$.

$y(1) = 1^2 - 4\ln(1+1) - 1 = 1 - 4\ln 2 - 1 = \boxed{-4\ln 2}$

c) Speed is increasing when v is moving away from zero. In other words, either $v>0$ and $v'>0$, or $v<0$ and $v'<0$.

$v'(t) = 2 + \frac{4}{(t+1)^2}$, which is always positive.

From above, $v>0$ on $t \in (1, \infty)$.

Therefore speed is increasing on $\boxed{(1, \infty)}$.

d) Total distance

$$y(0) = 0 - 4\ln 1 - 1 = -1$$
$$y(1) = 1 - 4\ln 2 - 1 = -4\ln 2$$

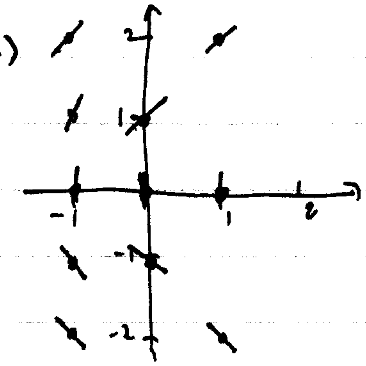
distance $= -1 - (-4\ln 2)$
$= -1 + 4\ln 2$

$$y(1) = -4\ln 2$$
$$y(2) = 4 - 4\ln 3 - 1 = 3 - 4\ln 3$$

distance $= 3 - 4\ln 3 - (-4\ln 2)$
$= 3 - 4\ln 3 + 4\ln 2$

$$\text{Total} = (-1 + 4\ln 2) + (3 - 4\ln 3 + 4\ln 2)$$
$$= 2 + 8\ln 2 - 4\ln 3 = \boxed{2 + \ln\left(\frac{2^8}{3^4}\right)}$$

5. $\dfrac{dy}{dx} = \dfrac{1+x^2}{y}$

a) 

Undefined when $y=0$ ⇒ no slopes there, but vertical tangent lines

$(-1,2)$ and $(1,2) \Rightarrow m = 1$

$(-1,1) \Rightarrow m = 2$

$(0,1) \Rightarrow m = 1$

$(-1,-1) \Rightarrow m = -2$ $\quad (0,-1) \Rightarrow m = -1$

$(-1,-2)$ and $(1,-2) \Rightarrow m = -1$

b) ~~As $y \to 0^+$, the slopes get steeper,~~

At $x=0$, the numerator of the slope is 1, so $y'(0)$ cannot be 0, as in the graph shown.

c) $\int y \, dy = \int (1+x^2)\,dx$

$\frac{1}{2}y^2 = x + \frac{1}{3}x^3 + C$

$y^2 = 2x + \frac{2}{3}x^3 + C$

$y = -\sqrt{2x + \frac{2}{3}x^3 + C}$

$y(3) = -4 \Rightarrow y < 0$

$-4 = -\sqrt{2\cdot 3 + \frac{2}{3}\cdot 3^3 + C}$

$16 = 6 + 18 + C$

$-8 = C$

$\therefore \boxed{y = -\sqrt{2x + \frac{2}{3}x^3 - 8}}$

6. Volume of oil $= 100{,}000\, m^3$

$\frac{dr}{dt} = 3$ m/min At $t=T$, area $= 100\pi m^2$

~~a~~ a) find $\frac{dA}{dt}$ at $t=T$.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

At $t=T$, $100\pi = \pi r^2 \Rightarrow r = 10$.

$\frac{dA}{dt} = 2\pi \cdot 10 \cdot 3 = \boxed{60\pi \ m^2/min}$

b) Thickness of oil, $y(t) = \frac{100000}{\pi r^2} = \frac{100000}{\pi} r^{-2}$

$\frac{dy}{dt} = -\frac{200000}{\pi} r^{-3} \cdot \frac{dr}{dt}$

At $t=T$, $\frac{dy}{dt} = \frac{-200000}{\pi} \cdot \frac{1}{1000} \cdot 3 = \boxed{\frac{-600}{\pi} \ m/min}$

c) Rate of change of area with respect to thickness.

$y = \frac{100000}{A} = 100000 A^{-1}$, so

~~$\frac{dy}{dA}$~~ $A = 100000 y^{-1}$

$\frac{dA}{dy} = -100000 y^{-2} \cdot$ ~~$\frac{dy}{dt}$~~

At $t=T$, $y = \frac{100000}{\pi \cdot 100} = \frac{1000}{\pi}$

$\frac{dA}{dt} = \frac{-100000}{\left(\frac{1000}{\pi}\right)^2} = \frac{-100000}{1000000} \cdot \pi^2 = \boxed{\frac{-1}{10}\pi^2 \ m^2/m}$