Course Framework Components

Overview

This course framework provides a clear and detailed description of the course requirements necessary for student success. The framework specifies what students should know, be able to do, and understand to qualify for college credit or placement.

The course framework includes two essential components:

① MATHEMATICAL PRACTICES

The mathematical practices are central to the study and practice of calculus. Students should develop and apply the described skills on a regular basis over the span of the course.

② COURSE CONTENT

The course content is organized into commonly taught units of study that provide a suggested sequence for the course. These units comprise the content and conceptual understandings that colleges and universities typically expect students to master to qualify for college credit and/or placement. This content is grounded in big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course.

1

AP CALCULUS AB AND BC

Mathematical Practices

The AP Calculus AB and BC mathematical practices describe what a student should be able to do while exploring course concepts. The table that follows presents these practices, which students should develop during the AP Calculus AB and AP Calculus BC courses. These practices are categorized into skills, which form the basis of the tasks on the AP Exam.

The unit guides later in this publication embed and spiral these skills throughout the course, providing teachers with one way to integrate the skills in the course content with sufficient repetition to prepare students to transfer those skills when taking the AP Exam. Course content may be paired with a variety of skills on the AP Exam.

More detailed information about teaching the mathematical practices can be found in the Instructional Approaches section of this publication.

AP CALCULUS AB AND BC

Mathematical Practices

Practice 1	Practice 2	Practice 3	Practice 4
Implementing Mathematical Processes **1**	***Connecting Representations*** **2**	***Justification*** **3**	***Communication and Notation*** **4**
Determine expressions and values using mathematical procedures and rules.	Translate mathematical information from a single representation or across multiple representations.	Justify reasoning and solutions.	Use correct notation, language, and mathematical conventions to communicate results or solutions.
SKILLS			
1.A Identify the question to be answered or problem to be solved *(not assessed)*.	**2.A** Identify common underlying structures in problems involving different contextual situations.	**3.A** Apply technology to develop claims and conjectures *(not assessed)*.	**4.A** Use precise mathematical language.
1.B Identify key and relevant information to answer a question or solve a problem *(not assessed)*.	**2.B** Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	**3.B** Identify an appropriate mathematical definition, theorem, or test to apply.	**4.B** Use appropriate units of measure.
1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., *Use the chain rule to find the derivative of a composite function*).	**2.C** Identify a re-expression of mathematical information presented in a given representation.	**3.C** Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	**4.C** Use appropriate mathematical symbols and notation (e.g., *Represent a derivative using* $f'(x)$, y', *and* $\frac{dy}{dx}$).
1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., *rate of change and accumulation*) or processes (e.g., *differentiation and its inverse process, anti-differentiation*) to solve problems.	**2.D** Identify how mathematical characteristics or properties of functions are related in different representations.	**3.D** Apply an appropriate mathematical definition, theorem, or test.	**4.D** Use appropriate graphing techniques.
1.E Apply appropriate mathematical rules or procedures, with and without technology.	**2.E** Describe the relationships among different representations of functions and their derivatives.	**3.E** Provide reasons or rationales for solutions and conclusions.	**4.E** Apply appropriate rounding procedures.
1.F Explain how an approximated value relates to the actual value.		**3.F** Explain the meaning of mathematical solutions in context.	
		3.G Confirm that solutions are accurate and appropriate.	

2

AP CALCULUS AB AND BC

Course Content

Based on the Understanding by Design® (Wiggins and McTighe) model, this course framework provides a clear and detailed description of the course requirements necessary for student success. The framework specifies what students must know, be able to do, and understand, with a focus on big ideas that encompass core principles, theories, and processes of the discipline. The framework also encourages instruction that prepares students for advanced coursework in mathematics or other fields engaged in modeling change (e.g., pure sciences, engineering, or economics) and for creating useful, reasonable solutions to problems encountered in an ever-changing world.

Big Ideas

The big ideas serve as the foundation of the course and allow students to create meaningful connections among concepts. They are often abstract concepts or themes that become threads that run throughout the course. Revisiting the big ideas and applying them in a variety of contexts allows students to develop deeper conceptual understanding. Below are the big ideas of the course and a brief description of each.

BIG IDEA 1: CHANGE (CHA)

Using derivatives to describe rates of change of one variable with respect to another or using definite integrals to describe the net change in one variable over an interval of another allows students to understand change in a variety of contexts. It is critical that students grasp the relationship between integration and differentiation as expressed in the Fundamental Theorem of Calculus—a central idea in AP Calculus.

BIG IDEA 2: LIMITS (LIM)

Beginning with a discrete model and then considering the consequences of a limiting case allows us to model real-world behavior and to discover and understand important ideas, definitions, formulas, and theorems in calculus: for example, continuity, differentiation, integration, and series **BC ONLY**.

BIG IDEA 3: ANALYSIS OF FUNCTIONS (FUN)

Calculus allows us to analyze the behaviors of functions by relating limits to differentiation, integration, and infinite series and relating each of these concepts to the others.

UNITS

The course content is organized into commonly taught units. The units have been arranged in a logical sequence frequently found in many college courses and textbooks.

The eight units in AP Calculus AB and ten units in AP Calculus BC, and their weighting on the multiple-choice section of the AP Exam, are listed on the following page.

Pacing recommendations at the unit level and on the Course at a Glance provide suggestions for how teachers can teach the required course content and administer the Personal Progress Checks.

The suggested class periods are based on a schedule in which the class meets five days a week for 45 minutes each day. While these recommendations have been made to aid planning, teachers are of course free to adjust the pacing based on the needs of their students, alternate schedules (e.g., block scheduling), or their school's academic calendar.

TOPICS

Each unit is broken down into teachable segments called topics. The topic pages (starting on p. 35) contain the required content for each topic. Although most topics can be taught in one or two class periods, teachers should pace the course to suit the needs of their students and school.

Exam Weighting for the Multiple-Choice Section of the AP Exam

Units	Exam Weighting (AB)	Exam Weighting (BC)
Unit 1: Limits and Continuity	**10–12%**	**4–7%**
Unit 2: Differentiation: Definition and Fundamental Properties	**10–12%**	**4–7%**
Unit 3: Differentiation: Composite, Implicit, and Inverse Functions	**9–13%**	**4–7%**
Unit 4: Contextual Applications of Differentiation	**10–15%**	**6–9%**
Unit 5: Analytical Applications of Differentiation	**15–18%**	**8–11%**
Unit 6: Integration and Accumulation of Change	**17–20%**	**17–20%**
Unit 7: Differential Equations	**6–12%**	**6–9%**
Unit 8: Applications of Integration	**10–15%**	**6–9%**
Unit 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions **BC ONLY**		**11–12%**
Unit 10: Infinite Sequences and Series **BC ONLY**		**17–18%**

Spiraling the Big Ideas

The following table shows how the big ideas spiral across units.

Big Ideas	**Unit 1** Limits and Continuity	**Unit 2** Differentiation: Definition and Fundamental Properties	**Unit 3** Differentiation: Composite, Implicit, and Inverse Functions	**Unit 4** Contextual Applications of Differentiation	**Unit 5** Analytical Applications of Differentiation
Change **CHA**	✓	✓		✓	
Limits **LIM**	✓	✓		✓	
Analysis of Functions **FUN**	✓	✓	✓		✓

Return to Table of Contents

Spiraling the Big Ideas *(cont'd)*

Big Ideas	Unit 6 Integration and Accumulation of Change	Unit 7 Differential Equations	Unit 8 Applications of Integration	Unit 9 Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY	Unit 10 Infinite Sequences and Series BC ONLY
Change **CHA**	✓		✓	✓	
Limits **LIM**	✓				✓
Analysis of Functions **FUN**	✓	✓		✓	

Course at a Glance

Plan

The Course at a Glance provides a useful visual organization of the AP Calculus AB and AP Calculus BC curricular components, including:

- Sequence of units, along with approximate weighting and suggested pacing. Please note, pacing is based on 45-minute class periods, meeting five days each week for a full academic year.
- Progression of topics within each unit.
- Spiraling of the big ideas and mathematical practices across units.

Teach

MATHEMATICAL PRACTICES

Mathematical practices spiral throughout the course.

1 Implementing Mathematical Processes	3 Justification
2 Connecting Representations	4 Communication and Notation

BIG IDEAS

Big ideas spiral across topics and units.

CHA Change	FUN Analysis of Functions
LIM Limits	

BC ONLY

The purple shading represents BC only content.

Assess

Assign the Personal Progress Checks—either as homework or in class—for each unit. Each Personal Progress Check contains formative multiple-choice and free-response questions. The feedback from the Personal Progress Checks shows students the areas where they need to focus.

UNIT 1 Limits and Continuity

AP EXAM WEIGHTING **10–12%** AB **4–7%** BC

CLASS PERIODS **~22–23** AB **~13–14** BC

CHA	2	1.1	Introducing Calculus: Can Change Occur at an Instant?
LIM	2	1.2	Defining Limits and Using Limit Notation
LIM	2	1.3	Estimating Limit Values from Graphs
LIM	2	1.4	Estimating Limit Values from Tables
LIM	1	1.5	Determining Limits Using Algebraic Properties of Limits
LIM	1	1.6	Determining Limits Using Algebraic Manipulation
LIM	1	1.7	Selecting Procedures for Determining Limits
LIM	3	1.8	Determining Limits Using the Squeeze Theorem
LIM	2	1.9	Connecting Multiple Representations of Limits
LIM	3	1.10	Exploring Types of Discontinuities
LIM	3	1.11	Defining Continuity at a Point
LIM	1	1.12	Confirming Continuity over an Interval
LIM	1	1.13	Removing Discontinuities
LIM	3	1.14	Connecting Infinite Limits and Vertical Asymptotes
LIM	2	1.15	Connecting Limits at Infinity and Horizontal Asymptotes
FUN	3	1.16	Working with the Intermediate Value Theorem (IVT)

Personal Progress Check 1

Multiple-choice: ~45 questions

Free-response: 3 questions (partial)

UNIT 2 Differentiation: Definition and Basic Derivative Rules

AP EXAM WEIGHTING **10–12%** AB **4–7%** BC

CLASS PERIODS **~13–14** AB **~9–10** BC

CHA	2	2.1	Defining Average and Instantaneous Rates of Change at a Point
CHA	1 4	2.2	Defining the Derivative of a Function and Using Derivative Notation
CHA	1	2.3	Estimating Derivatives of a Function at a Point
FUN	3	2.4	Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist
FUN	1	2.5	Applying the Power Rule
FUN	1	2.6	Derivative Rules: Constant, Sum, Difference, and Constant Multiple
FUN LIM	1	2.7	Derivatives of cos x, sin x, e^x, and ln x
FUN	1	2.8	The Product Rule
FUN	1	2.9	The Quotient Rule
FUN	1	2.10	Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

Personal Progress Check 2

Multiple-choice: ~30 questions

Free-response: 3 questions (partial)

NOTE: Partial versions of the free-response questions are provided to prepare students for more complex, full questions that they will encounter on the AP Exam.

UNIT 3 Differentiation: Composite, Implicit, and Inverse Functions

AP EXAM WEIGHTING 9–13% AB 4–7% BC

CLASS PERIODS ~10–11 AB ~8–9 BC

FUN	1	**3.1** The Chain Rule
FUN	1	**3.2** Implicit Differentiation
FUN	3	**3.3** Differentiating Inverse Functions
FUN	1	**3.4** Differentiating Inverse Trigonometric Functions
FUN	1	**3.5** Selecting Procedures for Calculating Derivatives
FUN	1	**3.6** Calculating Higher-Order Derivatives

Personal Progress Check 3

Multiple-choice: ~15 questions
Free-response: 3 questions
(partial/full)

UNIT 4 Contextual Applications of Differentiation

AP EXAM WEIGHTING 10–15% AB 6–9% BC

CLASS PERIODS ~10–11 AB ~6–7 BC

CHA	1	**4.1** Interpreting the Meaning of the Derivative in Context
CHA	1	**4.2** Straight-Line Motion: Connecting Position, Velocity, and Acceleration
CHA	2	**4.3** Rates of Change in Applied Contexts Other Than Motion
CHA	1	**4.4** Introduction to Related Rates
CHA	3	**4.5** Solving Related Rates Problems
CHA	1	**4.6** Approximating Values of a Function Using Local Linearity and Linearization
LIM	3	**4.7** Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

Personal Progress Check 4

Multiple-choice: ~15 questions
Free-response: 3 questions

UNIT 5 Analytical Applications of Differentiation

AP EXAM WEIGHTING 15–18% AB 8–11% BC

CLASS PERIODS ~15–16 AB ~10–11 BC

FUN	3	**5.1** Using the Mean Value Theorem
FUN	3	**5.2** Extreme Value Theorem, Global Versus Local Extrema, and Critical Points
FUN	2	**5.3** Determining Intervals on Which a Function Is Increasing or Decreasing
FUN	3	**5.4** Using the First Derivative Test to Determine Relative (Local) Extrema
FUN	1	**5.5** Using the Candidates Test to Determine Absolute (Global) Extrema
FUN	2	**5.6** Determining Concavity of Functions over Their Domains
FUN	3	**5.7** Using the Second Derivative Test to Determine Extrema
FUN	2	**5.8** Sketching Graphs of Functions and Their Derivatives
FUN	2	**5.9** Connecting a Function, Its First Derivative, and Its Second Derivative
FUN	2	**5.10** Introduction to Optimization Problems
FUN	3	**5.11** Solving Optimization Problems
FUN	1 3	**5.12** Exploring Behaviors of Implicit Relations

Personal Progress Check 5

Multiple-choice: ~35 questions
Free-response: 3 questions

UNIT 6
Integration and Accumulation of Change

AP EXAM WEIGHTING 17–20% AB 17–20% BC

CLASS PERIODS ~18–20 AB ~15–16 BC

CHA	4	**6.1**	Exploring Accumulations of Change
LIM	1	**6.2**	Approximating Areas with Riemann Sums
LIM	2	**6.3**	Riemann Sums, Summation Notation, and Definite Integral Notation
FUN	1	**6.4**	The Fundamental Theorem of Calculus and Accumulation Functions
FUN	2	**6.5**	Interpreting the Behavior of Accumulation Functions Involving Area
FUN	3	**6.6**	Applying Properties of Definite Integrals
FUN	3	**6.7**	The Fundamental Theorem of Calculus and Definite Integrals
FUN	4	**6.8**	Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation
FUN	1	**6.9**	Integrating Using Substitution
FUN	1	**6.10**	Integrating Functions Using Long Division and Completing the Square
FUN	1	**6.11**	Integrating Using Integration by Parts **BC ONLY**
FUN	1	**6.12**	Using Linear Partial Fractions **BC ONLY**
LIM	1	**6.13**	Evaluating Improper Integrals **BC ONLY**
FUN	1	**6.14**	Selecting Techniques for Antidifferentiation

Personal Progress Check 6

Multiple-choice:
- ~25 questions (AB)
- ~35 questions (BC)

Free-response: 3 questions

UNIT 7
Differential Equations

AP EXAM WEIGHTING 6–12% AB 6–9% BC

CLASS PERIODS ~8–9 AB ~9–10 BC

FUN	2	**7.1**	Modeling Situations with Differential Equations
FUN	3	**7.2**	Verifying Solutions for Differential Equations
FUN	2	**7.3**	Sketching Slope Fields
FUN	4	**7.4**	Reasoning Using Slope Fields
FUN	1	**7.5**	Approximating Solutions Using Euler's Method **BC ONLY**
FUN	1	**7.6**	Finding General Solutions Using Separation of Variables
FUN	1	**7.7**	Finding Particular Solutions Using Initial Conditions and Separation of Variables
FUN	3	**7.8**	Exponential Models with Differential Equations
FUN	3	**7.9**	Logistic Models with Differential Equations **BC ONLY**

Personal Progress Check 7

Multiple-choice:
- ~15 questions (AB)
- ~20 questions (BC)

Free-response: 3 questions

UNIT 8
Applications of Integration

AP EXAM WEIGHTING 10–15% AB 6–9% BC

CLASS PERIODS ~19–20 AB ~13–14 BC

CHA	1	**8.1**	Finding the Average Value of a Function on an Interval
CHA	1	**8.2**	Connecting Position, Velocity, and Acceleration of Functions Using Integrals
CHA	3	**8.3**	Using Accumulation Functions and Definite Integrals in Applied Contexts
CHA	4	**8.4**	Finding the Area Between Curves Expressed as Functions of *x*
CHA	1	**8.5**	Finding the Area Between Curves Expressed as Functions of *y*
CHA	2	**8.6**	Finding the Area Between Curves That Intersect at More Than Two Points
CHA	3	**8.7**	Volumes with Cross Sections: Squares and Rectangles
CHA	3	**8.8**	Volumes with Cross Sections: Triangles and Semicircles
CHA	3	**8.9**	Volume with Disc Method: Revolving Around the *x*- or *y*-Axis
CHA	2	**8.10**	Volume with Disc Method: Revolving Around Other Axes
CHA	4	**8.11**	Volume with Washer Method: Revolving Around the *x*- or *y*-Axis
CHA	2	**8.12**	Volume with Washer Method: Revolving Around Other Axes
CHA	3	**8.13**	The Arc Length of a Smooth, Planar Curve and Distance Traveled **BC ONLY**

Personal Progress Check 8

Multiple-choice: ~30 questions

Free-response: 3 questions

UNIT 9 Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY

AP EXAM WEIGHTING	N/A AB	11–12% BC
CLASS PERIODS	N/A AB	~10–11 BC

CHA	2	9.1	Defining and Differentiating Parametric Equations
CHA	1	9.2	Second Derivatives of Parametric Equations
CHA	1	9.3	Finding Arc Lengths of Curves Given by Parametric Equations
CHA	1	9.4	Defining and Differentiating Vector-Valued Functions
FUN	1	9.5	Integrating Vector-Valued Functions
FUN	1	9.6	Solving Motion Problems Using Parametric and Vector-Valued Functions
FUN	2	9.7	Defining Polar Coordinates and Differentiating in Polar Form
CHA	3	9.8	Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve
CHA	3	9.9	Finding the Area of the Region Bounded by Two Polar Curves

Personal Progress Check 9

Multiple-choice: ~25 questions
Free-response: 3 questions

UNIT 10 Infinite Sequences and Series BC ONLY

AP EXAM WEIGHTING	N/A AB	17–18% BC
CLASS PERIODS	N/A AB	~17–18 BC

LIM	3	10.1	Defining Convergent and Divergent Infinite Series
LIM	3	10.2	Working with Geometric Series
LIM	3	10.3	The nth Term Test for Divergence
LIM	3	10.4	Integral Test for Convergence
LIM	3	10.5	Harmonic Series and p-Series
LIM	3	10.6	Comparison Tests for Convergence
LIM	3	10.7	Alternating Series Test for Convergence
LIM	3	10.8	Ratio Test for Convergence
LIM	3	10.9	Determining Absolute or Conditional Convergence
LIM	1	10.10	Alternating Series Error Bound
LIM	3 2	10.11	Finding Taylor Polynomial Approximations of Functions
LIM	1	10.12	Lagrange Error Bound
LIM	2	10.13	Radius and Interval of Convergence of Power Series
LIM	2	10.14	Finding Taylor or Maclaurin Series for a Function
LIM	3	10.15	Representing Functions as Power Series

Personal Progress Check 10

Multiple-choice: ~45 questions
Free-response: 3 questions

AP CALCULUS AB AND BC

UNIT 1

Limits and Continuity

AP EXAM WEIGHTING	10–12% AB 4–7% BC
CLASS PERIODS	~22–23 AB ~13–14 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 1

Multiple-choice: ~45 questions
Free-response: 3 questions (partial)

UNIT
1

AP EXAM WEIGHTING	10–12% AB	4–7% BC
CLASS PERIODS	~22–23 AB	~13–14 BC

Limits and Continuity

BIG IDEA 1
***Change* CHA**
- Can change occur at an instant?

BIG IDEA 2
***Limits* LIM**
- How does knowing the value of a limit, or that a limit does not exist, help you to make sense of interesting features of functions and their graphs?

BIG IDEA 3
***Analysis of Functions* FUN**
- How do we close loopholes so that a conclusion about a function is always true?

Developing Understanding

Limits introduce the subtle distinction between evaluating a function at a point and considering what value the function is approaching, if any, as x approaches a point. This distinction allows us to extend understanding of asymptotes and holes in graphs with formal definitions of continuity. Consider reviewing rational functions when introducing limits, rather than beginning the year with a full review of precalculus topics. Limits are the foundation for differentiation (Unit 2), integration (Unit 6), and infinite series (Unit 10) **BC ONLY**. They are the basis for important definitions and for theorems that are used to solve realistic problems involving change and to justify conclusions.

Building the Mathematical Practices

2.B 2.C 3.C 3.D

Mathematical information may be organized or presented graphically, numerically, analytically, or verbally. Mathematicians must be able to communicate effectively in all of these contexts and transition seamlessly from one representation to another. Limits lay the groundwork for students' ongoing development of skills associated with taking what is presented in a table, an equation, or a sentence and translating that information into a graph (or vice versa). Help students explicitly practice matching different representations that show the same information, focusing on building their comfort level with translating analytical and verbal representations. This will be instrumental to their development of proficiency in this practice. The use of graphing calculators to help students explore these connections is strongly encouraged.

Mathematicians also explain reasoning and justify conclusions using definitions, theorems, and tests. A common student misunderstanding is that they don't need to write relevant given information before drawing the conclusion of a theorem.

In Unit 1, students should be given explicit instruction and time to practice "connecting the dots" by first demonstrating that all conditions or hypotheses have been met and then drawing the conclusion.

Preparing for the AP Exam

This course is a full-year experience building toward mastery assessed using the AP Exam. Therefore, it is important to consider both specific content and skills related to each unit and to build a coherent understanding of the whole. After studying Unit 1, students should be prepared to evaluate or estimate limits presented graphically, numerically, symbolically, or verbally. To avoid missed opportunities to earn points on the AP Exam, students should consistently practice using correct mathematical notation and presenting set-ups and appropriately rounded answers when using a calculator. Two sections of the exam do not allow calculator use. Some questions on the other two sections require it. From the first unit onward, emphasize the importance of hypotheses for theorems. Explore why each hypothesis is needed in order to ensure that the conclusion follows. Students should establish the practice of explicitly verifying that a theorem's hypotheses are satisfied before applying the theorem.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~22–23 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-1	**1.1 Introducing Calculus: Can Change Occur at an Instant?**	**2.B** Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.	
LIM-1	**1.2 Defining Limits and Using Limit Notation**	**2.B** Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.	
	1.3 Estimating Limit Values from Graphs	**2.B** Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.	
	1.4 Estimating Limit Values from Tables	**2.B** Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.	
	1.5 Determining Limits Using Algebraic Properties of Limits	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	1.6 Determining Limits Using Algebraic Manipulation	**1.C** Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	1.7 Selecting Procedures for Determining Limits	**1.C** Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	1.8 Determining Limits Using the Squeeze Theorem	**3.C** Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	
	1.9 Connecting Multiple Representations of Limits	**2.C** Identify a re-expression of mathematical information presented in a given representation.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~22–23 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
LIM-2	**1.10 Exploring Types of Discontinuities**	**3.B** Identify an appropriate mathematical definition, theorem, or test to apply.	
	1.11 Defining Continuity at a Point	**3.C** Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	
	1.12 Confirming Continuity over an Interval	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	1.13 Removing Discontinuities	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	1.14 Connecting Infinite Limits and Vertical Asymptotes	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
	1.15 Connecting Limits at Infinity and Horizontal Asymptotes	**2.D** Identify how mathematical characteristics or properties of functions are related in different representations.	
FUN-1	**1.16 Working with the Intermediate Value Theorem (IVT)**	**3.E** Provide reasons or rationales for solutions or conclusions.	
AP	Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 1. Review the results in class to identify and address any student misunderstandings.		

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	1.2	**Notation Read Aloud** Begin by writing a limit expression in analytical form (e.g., $\lim_{x \to 0^-} x^3$), and then read the expression aloud to the class: "The limit of x cubed as x approaches 0 from the left." Do the same for 1–2 additional examples that use a variety of limit notations (e.g., the symbol for infinity). Then have students pair up and take turns reading aloud different limit expressions to one another.
2	1.3 1.4	**Create Representations** Present students with a limit expression in analytical form (e.g., $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$), and then have them translate that expression into a variety of representations: constructing a graph, creating a table of values, and writing it as a verbal expression. Then have students check their graphs and tables using technology.
3	1.7	**Work Backward** Present students with a set of limit problems. Rather than determining the given limits, have them make a list of the various strategies that would be used to determine the limits (e.g., factoring, multiplying by conjugate, and simplify using trigonometric identities). After confirming their list is complete, have students work in pairs to create and write limit problems, each requiring one of the listed strategies. Then have them swap problems with another pair of students to complete each other's problems.
4	1.11	**Discussion Groups** Give each group of students a piecewise-defined function, a graph paper, and a list of x-values. Have them graph the function, then discuss whether the function is continuous or discontinuous at each x-value, and explain why. Ask students to take turns recording the group's conclusion for each x-value. If continuous, have students discuss and show that all three continuity conditions are satisfied. If discontinuous, have students state which condition was not satisfied.
5	1.16	**Think Aloud** In small groups, have students discuss the Intermediate Value Theorem and share ideas about real-world applications (e.g., speed of your car and weight of your kitten). Have groups post their ideas on a classroom wall using sticky notes.

TOPIC 1.1

Introducing Calculus: Can Change Occur at an Instant?

Required Course Content

ENDURING UNDERSTANDING

CHA-1

Calculus allows us to generalize knowledge about motion to diverse problems involving change.

LEARNING OBJECTIVE

CHA-1.A

Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

ESSENTIAL KNOWLEDGE

CHA-1.A.1

Calculus uses limits to understand and model dynamic change.

CHA-1.A.2

Because an average rate of change divides the change in one variable by the change in another, the average rate of change is undefined at a point where the change in the independent variable would be zero.

CHA-1.A.3

The limit concept allows us to define instantaneous rate of change in terms of average rates of change.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "say" some of the notation**

TOPIC 1.2

Defining Limits and Using Limit Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-1.A Represent limits analytically using correct notation.	**LIM-1.A.1** Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is $\lim_{x \to c} f(x) = R$. ☒ **EXCLUSION STATEMENT** *The epsilon-delta definition of a limit is not assessed on the AP Calculus AB or BC Exam. However, teachers may include this topic in the course if time permits.*
LIM-1.B Interpret limits expressed in analytic notation.	**LIM-1.B.1** A limit can be expressed in multiple ways, including graphically, numerically, and analytically.

TOPIC 1.3

Estimating Limit Values from Graphs

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.1

The concept of a limit includes one sided limits.

LIM-1.C.2

Graphical information about a function can be used to estimate limits.

LIM-1.C.3

Because of issues of scale, graphical representations of functions may miss important function behavior.

LIM-1.C.4

A limit might not exist for some functions at particular values of x. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

ILLUSTRATIVE EXAMPLES

For LIM-1.C.4:

- $\lim\limits_{x\to 0}\frac{1}{x^2}=\infty$
- $\lim\limits_{x\to 0}\frac{|x|}{x}$ does not exist.
- $\lim\limits_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.
- $\lim\limits_{x\to 0}\frac{1}{x}$ does not exist.

AVAILABLE RESOURCES

- **AP Calculus Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**
- Classroom Resource > **Approximation**

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**
- Classroom Resource > **Approximation**

TOPIC 1.4

Estimating Limit Values from Tables

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.5

Numerical information can be used to estimate limits.

TOPIC 1.5

Determining Limits Using Algebraic Properties of Limits

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-1.D Determine the limits of functions using limit theorems.	**LIM-1.D.1** One-sided limits can be determined analytically or graphically. **LIM-1.D.2** Limits of sums, differences, products, quotients, and composite functions can be found using limit theorems.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

ILLUSTRATIVE EXAMPLES

- Factoring and dividing common factors of rational functions
- Multiplying by an expression involving the conjugate of a sum or difference in order to simplify functions involving radicals
- Using alternate forms of trigonometric functions

TOPIC 1.6

Determining Limits Using Algebraic Manipulation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.1

It may be necessary or helpful to rearrange expressions into equivalent forms before evaluating limits.

TOPIC 1.7

Selecting Procedures for Determining Limits

This topic is intended to focus on the skill of selecting an appropriate procedure for determining limits. Students should be given opportunities to practice when and how to apply all learning objectives relating to determining limits.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 1.8

Determining Limits Using the Squeeze Theorem

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.2

The limit of a function may be found by using the squeeze theorem.

SUGGESTED SKILL

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

ILLUSTRATIVE EXAMPLES

The squeeze theorem can be used to show $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Limits Questions**

Return to Table of Contents

TOPIC 1.9

Connecting Multiple Representations of Limits

This topic is intended to focus on connecting representations. Students should be given opportunities to practice when and how to apply all learning objectives relating to limits and translating mathematical information from a single representation or across multiple representations.

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**

SUGGESTED SKILL

Justification

3.B

Identify an appropriate mathematical definition, theorem, or test to apply.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**

TOPIC 1.10

Exploring Types of Discontinuities

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.1

Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.

TOPIC 1.11

Defining Continuity at a Point

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.2

A function f is continuous at $x = c$ provided that $f(c)$ exists, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$.

SUGGESTED SKILL

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Video on Continuity**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Video on Continuity**

TOPIC 1.12

Confirming Continuity over an Interval

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.B

Determine intervals over which a function is continuous.

ESSENTIAL KNOWLEDGE

LIM-2.B.1

A function is continuous on an interval if the function is continuous at each point in the interval.

LIM-2.B.2

Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous on all points in their domains.

TOPIC 1.13

Removing Discontinuities

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.C

Determine values of x or solve for parameters that make discontinuous functions continuous, if possible.

ESSENTIAL KNOWLEDGE

LIM-2.C.1

If the limit of a function exists at a discontinuity in its graph, then it is possible to remove the discontinuity by defining or redefining the value of the function at that point, so it equals the value of the limit of the function as x approaches that point.

LIM-2.C.2

In order for a piecewise-defined function to be continuous at a boundary to the partition of its domain, the value of the expression defining the function on one side of the boundary must equal the value of the expression defining the other side of the boundary, as well as the value of the function at the boundary.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- The Exam > **2012 Exam, MCQ #9**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 1.14

Connecting Infinite Limits and Vertical Asymptotes

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.D

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE

LIM-2.D.1

The concept of a limit can be extended to include infinite limits.

LIM-2.D.2

Asymptotic and unbounded behavior of functions can be described and explained using limits.

TOPIC 1.15

Connecting Limits at Infinity and Horizontal Asymptotes

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.D

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE

LIM-2.D.3

The concept of a limit can be extended to include limits at infinity.

LIM-2.D.4

Limits at infinity describe end behavior.

LIM-2.D.5

Relative magnitudes of functions and their rates of change can be compared using limits.

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions or conclusions.

AVAILABLE RESOURCES

- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**
- Classroom Resource > **Why We Use Theorem in Calculus**

TOPIC 1.16

Working with the Intermediate Value Theorem (IVT)

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.A

Explain the behavior of a function on an interval using the Intermediate Value Theorem.

ESSENTIAL KNOWLEDGE

FUN-1.A.1

If f is a continuous function on the closed interval $[a, b]$ and d is a number between $f(a)$ and $f(b)$, then the Intermediate Value Theorem guarantees that there is at least one number c between a and b, such that $f(c) = d$.

AP CALCULUS AB AND BC

UNIT 2

Differentiation: Definition and Fundamental Properties

AP EXAM WEIGHTING	10–12% AB	4–7% BC
CLASS PERIODS	~13–14 AB	~9–10 BC

Return to Table of Contents

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 2

Multiple-choice: ~30 questions
Free-response: 3 questions (partial)

UNIT
2

AP EXAM WEIGHTING	10–12% AB	4–7% BC
CLASS PERIODS	~13–14 AB	~9–10 BC

Differentiation: Definition and Fundamental Properties

BIG IDEA 1
***Change* CHA**

- How can a state determine the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) based on a model for the number of graduates as a function of the state's education budget?

BIG IDEA 2
***Limits* LIM**

- Why do mathematical properties and rules for simplifying and evaluating limits apply to differentiation?

BIG IDEA 3
***Analysis of Functions* FUN**

- If you knew that the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) was a positive number, what might that tell you about the number of graduates at that level of investment?

Developing Understanding

Derivatives allow us to determine instantaneous rates of change. To develop understanding of how the definition of the derivative applies limits to average rates of change, create opportunities for students to explore average rates of change over increasingly small intervals. Graphing calculator explorations of how various operations affect slopes of tangent lines help students to make sense of basic rules and properties of differentiation. Encourage students to apply the order of operations as they select differentiation rules. Developing differentiation skills will allow students to model realistic instantaneous rates of change in Unit 4 and to analyze graphs in Unit 5.

Building the Mathematical Practices

1.E **2.B** **4.C**

Mathematicians know that a solution will only be as good as the procedure used to find it and that the difference between being correct and incorrect can often be traced to an arithmetic or procedural error. In other words, mathematicians know that the details matter. Students often find it difficult to apply mathematical procedures—including the rules of differentiation—with precision and accuracy. For example, students may drop important notation, such as a parenthesis, or misapply the product rule by taking the derivative of each factor separately and then multiplying those together. The content of Unit 2 is a foundational entry point for practicing the skill of applying mathematical procedures and learning to self-correct before common mistakes occur.

This is also an opportunity to revisit and reinforce the practice of connecting representations, as students will be seeing derivatives presented in analytical, numerical, graphical, and verbal representations. Students can practice by extracting information about the original function, f, from a graphical representation of f'. This can help prevent misunderstandings when examining the graph of a derivative (such as misinterpreting it as the graph of the original function instead).

Preparing for the AP Exam

Students should practice presenting clear mathematical structures that connect their work with definitions or theorems. For example, when asked to estimate the slope of the line tangent to a curve at a given point based on information provided in a table of values, as in **2013 AP Exam Free-Response Question 3 Part A**, students must present a difference quotient:

$$C'(3.5) \approx \frac{C(4)-C(3)}{4-3} = \frac{12.8-11.2}{1}.$$

Failure to present this structure will cost students the point they might have earned, even with a correct numerical answer. Similarly, to evaluate the derivative of $f(x) = u(x) \cdot v(x)$ at $x = 3$, students should show the product rule structure, as in $f'(3) = u(3)v'(3) + v(3)u'(3)$, and import values. Finally, students should present mathematical expressions evaluated on the calculator and use specified rounding procedures, typically rounding or truncating to three places after the decimal point. It is helpful to establish the habit of storing intermediate calculations in the calculator in order to avoid accumulation of rounding errors.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
CHA-2	**2.1** Defining Average and Instantaneous Rates of Change at a Point	**2.B** Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.	
	2.2 Defining the Derivative of a Function and Using Derivative Notation	**1.D** Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems. **4.C** Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	2.3 Estimating Derivatives of a Function at a Point	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-2	**2.4** Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist	**3.E** Provide reasons or rationales for solutions and conclusions.	
FUN-3	**2.5** Applying the Power Rule	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3 LIM-3	**2.7** Derivatives of $\cos x$, $\sin x$, e^x, and $\ln x$	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
FUN-3	**2.8 The Product Rule**	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	2.9 The Quotient Rule	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions	**1.D** Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 2. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	2.1 2.2 2.3	**Graph and Switch** Present students with two or three functions and the graph of each function. Have each student choose a random derivative question and one function. Questions could include: Find the average rate of change on an interval, instantaneous rate of change at a point, derivative as a function, derivative value at a point, or equations for tangent or normal lines at a point. Have students answer their question and place their answer onto the function's graph. Then have students share their solutions with each other to give and receive feedback.
2	2.4	**Match Mine** Create cards containing graph images of functions with various continuous, discontinuous, differentiable, and nondifferentiable points or intervals. Provide each student in a pair with the same nine cards. Student A arranges their graphs in a 3 × 3 grid, which is not visible to Student B. Student A describes each of their graph's positions using information about continuity and differentiability to describe the graph. Based on the descriptions, Student B attempts to arrange their cards to match the grid of Student A.
3	2.5 2.6 2.7 2.8 2.9 2.10	**Error Analysis** Assign a function to each student. Ask them to find the function's derivative using one or more derivative rules. Allow them to check their answers. Ask half of the class to redo their work to include an error, thus having the wrong answer. Ask students to record their correct or incorrect work on a card. Mix up the cards and redistribute, having students determine if the answer is correct or incorrect. If incorrect, they should explain what error was made, and find the correct answer.
4	2.5 2.6 2.7 2.8 2.9 2.10	**Graphic Organizer** Provide students with colored paper, pens, and markers. Ask them to create a chart, a foldable card, or other creative method to organize all the derivative rules. For each rule, have them include the mathematical definition, examples, pictures, and helpful hints to understand and remember the rule.
5	2.8 2.9	**Round Table** Provide each student with the same worksheet containing four functions that require the product rule or quotient rule when finding the derivative. Then have students sit in groups of four. Each student determines the derivative of function No. 1, and then they pass their papers clockwise to the next student. Each student checks the first problem and, if necessary, discusses any mistakes with the previous student. Each student now completes function No. 2 on the paper, and the process continues until each student has their original paper back.

TOPIC 2.1

Defining Average and Instantaneous Rates of Change at a Point

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.A

Determine average rates of change using difference quotients.

ESSENTIAL KNOWLEDGE

CHA-2.A.1

The difference quotients $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(x)-f(a)}{x-a}$ express the average rate of change of a function over an interval.

CHA-2.B

Represent the derivative of a function as the limit of a difference quotient.

CHA-2.B.1

The instantaneous rate of change of a function at $x=a$ can be expressed by $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ or $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$, provided the limit exists. These are equivalent forms of the definition of the derivative and are denoted $f'(a)$.

SUGGESTED SKILLS

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "Say" Some of the Notation**

TOPIC 2.2

Defining the Derivative of a Function and Using Derivative Notation

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
CHA-2.B Represent the derivative of a function as the limit of a difference quotient.	**CHA-2.B.2** The derivative of f is the function whose value at x is $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided this limit exists. **CHA-2.B.3** For $y=f(x)$, notations for the derivative include $\frac{dy}{dx}$, $f'(x)$, and y'. **CHA-2.B.4** The derivative can be represented graphically, numerically, analytically, and verbally.
CHA-2.C Determine the equation of a line tangent to a curve at a given point.	**CHA-2.C.1** The derivative of a function at a point is the slope of the line tangent to a graph of the function at that point.

Return to Table of Contents

TOPIC 2.3

Estimating Derivatives of a Function at a Point

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Classroom Resource > **Approximation**
- Classroom Resource > **Reasoning from Tabular Data**

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.D

Estimate derivatives.

ESSENTIAL KNOWLEDGE

CHA-2.D.1

The derivative at a point can be estimated from information given in tables or graphs.

CHA-2.D.2

Technology can be used to calculate or estimate the value of a derivative of a function at a point.

TOPIC 2.4

Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

ILLUSTRATIVE EXAMPLES

For FUN-2.A.2:

- The left hand and right hand limits of the difference quotient are not equal, as in $f(x)=|x|$ at $x=0$.
- The tangent line is vertical and has no slope, as in $f(x)=\sqrt[3]{x}$ at $x=0$.

Required Course Content

ENDURING UNDERSTANDING

FUN-2

Recognizing that a function's derivative may also be a function allows us to develop knowledge about the related behaviors of both.

LEARNING OBJECTIVE

FUN-2.A

Explain the relationship between differentiability and continuity.

ESSENTIAL KNOWLEDGE

FUN-2.A.1

If a function is differentiable at a point, then it is continuous at that point. In particular, if a point is not in the domain of f, then it is not in the domain of f'.

FUN-2.A.2

A continuous function may fail to be differentiable at a point in its domain.

TOPIC 2.5

Applying the Power Rule

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.1

Direct application of the definition of the derivative and specific rules can be used to calculate the derivative for functions of the form $f(x)=x^r$.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 2.6

Derivative Rules: Constant, Sum, Difference, and Constant Multiple

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.2

Sums, differences, and constant multiples of functions can be differentiated using derivative rules.

FUN-3.A.3

The power rule combined with sum, difference, and constant multiple properties can be used to find the derivatives for polynomial functions.

TOPIC 2.7

Derivatives of cos x, sin x, e^x, and ln x

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.4

Specific rules can be used to find the derivatives for sine, cosine, exponential, and logarithmic functions.

ENDURING UNDERSTANDING

LIM-3

Reasoning with definitions, theorems, and properties can be used to determine a limit.

LEARNING OBJECTIVE

LIM-3.A

Interpret a limit as a definition of a derivative.

ESSENTIAL KNOWLEDGE

LIM-3.A.1

In some cases, recognizing an expression for the definition of the derivative of a function whose derivative is known offers a strategy for determining a limit.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 2.8

The Product Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.1

Derivatives of products of differentiable functions can be found using the product rule.

TOPIC 2.9

The Quotient Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.2

Derivatives of quotients of differentiable functions can be found using the quotient rule.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Selecting Procedures for Derivatives**
- AP Online Teacher Community Discussion > **Simplifying the Quotient Rule**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

TOPIC 2.10

Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.3

Rearranging tangent, cotangent, secant, and cosecant functions using identities allows differentiation using derivative rules.

AP CALCULUS AB AND BC

UNIT 3

Differentiation: Composite, Implicit, and Inverse Functions

AP EXAM WEIGHTING	9–13% AB 4–7% BC	
CLASS PERIODS	~10–11 AB ~8–9 BC	

Return to Table of Contents

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 3

Multiple-choice: ~15 questions
Free-response: 3 questions (partial/full)

UNIT
3

AP EXAM WEIGHTING	9–13% AB	4–7% BC
CLASS PERIODS	~10–11 AB	~8–9 BC

Differentiation: Composite, Implicit, and Inverse Functions

BIG IDEA 3
Analysis of Functions **FUN**

- If pressure experienced by a diver is a function of depth and depth is a function of time, how might we find the rate of change in pressure with respect to time?

Developing Understanding

In this unit, students learn how to differentiate composite functions using the chain rule and apply that understanding to determine derivatives of implicit and inverse functions. Students need to understand that for composite functions, y is a function of u while u is a function of x. Leibniz notation for the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, accounts for these relationships. Units analysis can strengthen the connection, as in $\frac{\text{psi}}{\text{min}} = \frac{\text{psi}}{\text{m}} \cdot \frac{\text{m}}{\text{min}}$. Saying, “times the derivative of what’s inside,” every time we apply the chain rule reminds students to avoid a common error. Mastering the chain rule is essential to success in all future units.

Building the Mathematical Practices

1.C 1.E 3.G

Identifying composite and implicit functions is a key differentiation skill. Students must recognize functions embedded in functions and be able to decompose composite functions into their “outer” and “inner” component functions. Misapplying the chain rule by forgetting to also differentiate the “inner” function or misidentifying the “inner” function are common errors. Provide sample responses that demonstrate these errors to help students be mindful of them in their own work. Reinforcing the chain rule structure sets the stage for Unit 6, when students learn the inverse of this process.

Students should continue to practice using correct notation and applying procedures accurately. Checking one another’s work, reviewing sample responses (with and without errors), and using technology to check calculations develop these skills. Emphasize that taking higher-order derivatives mirrors familiar differentiation processes (i.e., “function is to first derivative as first derivative is to second derivative”).

Use questioning techniques such as, “What does this mean?” to help students develop a more solid conceptual understanding of higher-order differentiation.

Preparing for the AP Exam

Mastery of the chain rule and its applications is essential for success on the AP Exam. The chain rule will be the target of assessment for many questions and a necessary step along the way for others. One common error is not recognizing when the chain rule applies, especially in composite functions such as $\sin^2 x$, $\tan(2x - 1)$, and e^{x^2}. In expressions like $\frac{y}{3y^2 - x}$, students must recognize that the chain rule applies to y because y depends on x. When multiple rules apply, students may struggle with the order of operations. Offer mixed practice differentiating general functions using select values provided in tables and graphs. Focus on products, quotients, compositions, and inverses of functions, especially those with names other than f and g. Connecting graphs, tables, and algebraic reasoning builds understanding of differentiation of inverse functions.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods **~10–11** CLASS PERIODS (AB) **~8–9** CLASS PERIODS (BC)
FUN-3	**3.1 The Chain Rule**	**1.C** Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.2 Implicit Differentiation	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	3.3 Differentiating Inverse Functions	**3.G** Confirm that solutions are accurate and appropriate.	
	3.4 Differentiating Inverse Trigonometric Functions	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	3.5 Selecting Procedures for Calculating Derivatives	**1.C** Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.6 Calculating Higher-Order Derivatives	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 3.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	3.1	**Scavenger Hunt** Place a card with a starter question somewhere in the classroom, for example, "Find the derivative of $f(x) = \sin(4x)$." Place another card in the room with the solution to that card plus another question: "Solution: $4\cos(4x)$. Next problem: Find the derivative of $f(x) = (\sin(x))^4$." Continue posting solution cards with new problems until the final card presents a problem whose solution is on the original starter card (note that this solution should be added to the starter card above).
2	3.1	**Work Backward** Provide a chain rule problem with several possible answers. Have students identify which piece of the original problem contributed its derivative to one of the factors in the answer they are looking at. For each possible answer, students should say whether the answer is correct. They should also identify which piece of the composition was differentiated incorrectly or skipped, or state which factor of that particular answer did not come from the original problem. Students often have trouble knowing when to stop differentiating when learning e^x or trigonometric functions, so make sure you include these types of problems.
3	3.2	**Round Table** In groups of four, each student has an identical paper with four different problems on it. Students complete the first problem on their paper and then pass the paper clockwise to another member in their group. That student checks the first problem and then completes the second problem on the paper. Students rotate again and the process continues until each student has their own paper back.
4	3.4	**Quiz-Quiz-Trade** Give students a card containing a question and have them write the answer on the back. Students then circulate around the room and find a partner. One student quizzes the other by showing only the side of the card with the question on it, and then they reverse roles. They swap cards, find a new partner, quiz each other, and the process continues.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C
Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 3.1
The Chain Rule

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.C

Calculate derivatives of compositions of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.C.1

The chain rule provides a way to differentiate composite functions.

TOPIC 3.2

Implicit Differentiation

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
FUN-3.D Calculate derivatives of implicitly defined functions.	**FUN-3.D.1** The chain rule is the basis for implicit differentiation.

SUGGESTED SKILL

Justification

3.G

Confirm that solutions are accurate and appropriate.

TOPIC 3.3

Differentiating Inverse Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.E

Calculate derivatives of inverse and inverse trigonometric functions.

ESSENTIAL KNOWLEDGE

FUN-3.E.1

The chain rule and definition of an inverse function can be used to find the derivative of an inverse function, provided the derivative exists.

TOPIC 3.4

Differentiating Inverse Trigonometric Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.E

Calculate derivatives of inverse and inverse trigonometric functions.

ESSENTIAL KNOWLEDGE

FUN-3.E.2

The chain rule applied with the definition of an inverse function, or the formula for the derivative of an inverse function, can be used to find the derivatives of inverse trigonometric functions.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

TOPIC 3.5

Selecting Procedures for Calculating Derivatives

This topic is intended to focus on the skill of selecting an appropriate procedure for calculating derivatives. Students should be given opportunities to practice when and how to apply all learning objectives relating to calculating derivatives.

TOPIC 3.6

Calculating Higher - Order Derivatives

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.F

Determine higher order derivatives of a function.

ESSENTIAL KNOWLEDGE

FUN-3.F.1

Differentiating f' produces the second derivative f'', provided the derivative of f' exists; repeating this process produces higher-order derivatives of f.

FUN-3.F.2

Higher-order derivatives are represented with a variety of notations. For $y = f(x)$, notations for the second derivative include $\frac{d^2y}{dx^2}$, $f''(x)$, and y''. Higher-order derivatives can be denoted $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

THIS PAGE IS INTENTIONALLY LEFT BLANK.

AP CALCULUS AB AND BC

UNIT 4

Contextual Applications of Differentiation

AP EXAM WEIGHTING	10–15% AB 6–9% BC	
CLASS PERIODS	~10–11 AB ~6–7 BC	

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 4

Multiple-choice: ~15 questions
Free-response: 3 questions

UNIT
4

AP EXAM WEIGHTING	**10–15%** AB	**6–9%** BC
CLASS PERIODS	**~10–11** AB	**~6–7** BC

Contextual Applications of Differentiation

BIG IDEA 1
Change **CHA**

- How are problems about position, velocity, and acceleration of a particle in motion over time structurally similar to problems about the volume of a rising balloon over an interval of heights, the population of London over the 14th century, or the metabolism of a dose of medicine over time?

BIG IDEA 2
Limits **LIM**

- Since certain indeterminate forms seem to actually approach a limit, how can we determine that limit, provided it exists?

Developing Understanding

Unit 4 begins by developing understanding of average and instantaneous rates of change in problems involving motion. The unit then identifies differentiation as a common underlying structure on which to build understanding of change in a variety of contexts. Students' understanding of units of measure often reinforces their understanding of contextual applications of differentiation. In problems involving related rates, identifying the independent variable common to related functions may help students to correctly apply the chain rule. When applying differentiation to determine limits of certain indeterminate forms using L'Hospital's rule, students must show that the rule applies.

Building the Mathematical Practices

1.D **1.F** **2.A** **3.F**

Students will begin applying concepts from Units 2 and 3 to scenarios encountered in the world. Students often struggle to translate these verbal scenarios into the mathematical procedures necessary to answer the question. To solve these problems, students will need explicit instruction and intentional practice identifying key information, determining which procedure applies to the scenario presented (i.e., that "rates of change" indicate differentiation), stating what is changing and how, using correct units, and explaining what their answer means in the context of the scenario. Provide scenarios with different contexts but similar procedures so students begin to recognize and apply the reasoning behind those problem-solving decisions, rather than grasping at rules haphazardly.

Students must also be able to explain how an approximated value relates to the value it's intended to approximate. Students may not understand why they would use a tangent line approximation (i.e., linearization) rather than simply evaluating a function. Expose them to scenarios where an exact function value can't be calculated, and then ask them to determine whether a particular approximation is an overestimate or an underestimate of the function.

Preparing for the AP Exam

With contextual problems, emphasize careful reading for language such as, "find the rate of change." This will help students understand the underlying structure of the problem, answer the question asked, and interpret solutions in context. Students should not use words like "velocity" when they mean the rate of change in income, for example, even though the underlying structure is the same.

Emphasize that students must verify that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (or that both approach infinity) as a necessary first step before applying L'Hospital's Rule to determine $\lim_{x\to a} \frac{f(x)}{g(x)}$. Students should understand that $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are appropriate labels for indeterminate forms but do not represent values in an equation. Therefore, it is incorrect to write $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$, for example. Note that $\lim_{x\to a} \frac{f(x)}{g(x)} \neq \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ when $\lim_{x\to a} g(x) = 0$. Also emphasize that the conclusion of L'Hospital's rule features the ratio of the derivatives of the numerator and denominator, respectively, rather than the derivative of the ratio.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~10–11 CLASS PERIODS (AB) ~6–7 CLASS PERIODS (BC)
CHA-3	**4.1 Interpreting the Meaning of the Derivative in Context**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	4.3 Rates of Change in Applied Contexts Other Than Motion	2.A Identify common underlying structures in problems involving different contextual situations.	
	4.4 Introduction to Related Rates	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	4.5 Solving Related Rates Problems	3.F Explain the meaning of mathematical solutions in context.	
	4.6 Approximating Values of a Function Using Local Linearity and Linearization	1.F Explain how an approximated value relates to the actual value.	
LIM-4	**4.7 Using L'Hospital's Rule for Determining Limits of Indeterminate Forms**	3.D Apply an appropriate mathematical definition, theorem, or test.	
	Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 4. Review the results in class to identify and address any student misunderstandings.		

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	4.1	**Quickwrite** Divide students into groups and give each group a context (outdoors, in a supermarket, in biology, in the government, at home, etc.). Students then write for a few minutes, listing things that are changing in that particular context.
2	4.2	**Create Representations** Provide verbal descriptions of a roller coaster ride: at time 0, velocity is 0 but about to become positive; at time 2, velocity is positive and increasing; at time 5, velocity is 0 and decreasing, etc. Have students graph position (from start), velocity, acceleration, speed, and then draw arrows at each point depicting whether their body would lean forward, backward, or not at all.
3	4.4	**Marking the Text** Have students read through a problem and highlight/underline the given quantities and directions in a problem, stating whether that information always applies or applies only at an instant.
4	4.5	**Round Table** Give students different related rates problems and a paper divided into five sections, titled as following: ▪ Draw a picture ▪ Equation ▪ Derivative ▪ Specific information used ▪ Interpretation Students first draw a picture of the situation and pass the papers clockwise. Students then critique the work in the previous section, complete the next section, and pass the papers again until all sections are completed.
5	4.6	**Scavenger Hunt** A starter question is posted in the room, for example, "Approximate the value . . . ." Have students work through the problem to find the value and then look for that value at the top of another card posted in the room. Students then solve the problem on that card, for example, "Write the equation of the tangent line . . . " and look for that solution on a third card, etc. The solution to the last problem will be on the starter card.

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

TOPIC 4.1

Interpreting the Meaning of the Derivative in Context

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
CHA-3.A Interpret the meaning of a derivative in context.	**CHA-3.A.1** The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable. **CHA-3.A.2** The derivative can be used to express information about rates of change in applied contexts. **CHA-3.A.3** The unit for $f'(x)$ is the unit for f divided by the unit for x.

TOPIC 4.2

Straight-Line Motion: Connecting Position, Velocity, and Acceleration

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.B

Calculate rates of change in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-3.B.1

The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

SUGGESTED SKILL

Connecting Representations

2.A

Identify common underlying structures in problems involving different contextual situations.

TOPIC 4.3

Rates of Change in Applied Contexts Other Than Motion

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
CHA-3.C Interpret rates of change in applied contexts.	**CHA-3.C.1** The derivative can be used to solve problems involving rates of change in applied contexts.

TOPIC 4.4

Introduction to Related Rates

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Related Rates: Analyzing Problems in Context**
- AP Online Teacher Community Discussion > **Related Rates in FRQ**

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.D

Calculate related rates in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-3.D.1

The chain rule is the basis for differentiating variables in a related rates problem with respect to the same independent variable.

CHA-3.D.2

Other differentiation rules, such as the product rule and the quotient rule, may also be necessary to differentiate all variables with respect to the same independent variable.

SUGGESTED SKILL

Justification

3.F

Explain the meaning of mathematical solutions in context.

AVAILABLE RESOURCES

- Professional Development > **Related Rates: Analyzing Problems in Context**
- AP Online Teacher Community Discussion > **Related Rates in FRQ**

TOPIC 4.5

Solving Related Rates Problems

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
CHA-3.E Interpret related rates in applied contexts.	**CHA-3.E.1** The derivative can be used to solve related rates problems; that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.

TOPIC 4.6

Approximating Values of a Function Using Local Linearity and Linearization

SUGGESTED SKILL

Implementing Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.F

Approximate a value on a curve using the equation of a tangent line.

ESSENTIAL KNOWLEDGE

CHA-3.F.1

The tangent line is the graph of a locally linear approximation of the function near the point of tangency.

CHA-3.F.2

For a tangent line approximation, the function's behavior near the point of tangency may determine whether a tangent line value is an underestimate or an overestimate of the corresponding function value.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- AP Online Teacher Community Discussion > **L'Hospital's Rule**
- AP Online Teacher Community Discussion > **Possible Inconsistent Language**
- The Exam > **2018 Chief Reader Report, FRQ #5(d)**
- The Exam > **2018 Samples and Commentary, FRQ #5(d)**
- The Exam > **2018 Scoring Guidelines, FRQ #5(d)**

TOPIC 4.7

Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

Required Course Content

ENDURING UNDERSTANDING

LIM-4

L'Hospital's Rule allows us to determine the limits of some indeterminate forms.

LEARNING OBJECTIVE

LIM-4.A

Determine limits of functions that result in indeterminate forms.

ESSENTIAL KNOWLEDGE

LIM-4.A.1

When the ratio of two functions tends to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in the limit, such forms are said to be indeterminate.

☒ EXCLUSION STATEMENT

There are many other indeterminate forms, such as $\infty - \infty$, for example, but these will not be assessed on either the AP Calculus AB or BC Exam. However, teachers may include these topics, if time permits.

LIM-4.A.2

Limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.

AP CALCULUS AB AND BC

UNIT 5

Analytical Applications of Differentiation

AP EXAM WEIGHTING	15–18% AB 8–11% BC	
CLASS PERIODS	~15–16 AB ~10–11 BC	

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 5

Multiple-choice: ~35 questions
Free-response: 3 questions

UNIT
5

AP EXAM WEIGHTING	15–18% AB	8–11% BC
CLASS PERIODS	~15–16 AB	~10–11 BC

Analytical Applications of Differentiation

BIG IDEA 3
Analysis of Functions **FUN**

- How might the Mean Value Theorem be used to justify a conclusion that you were speeding at some point on a certain stretch of highway, even without knowing the exact time you were speeding?
- What additional information is included in a sound mathematical argument about optimization that a simple description of an equivalent answer lacks?

Developing Understanding

In this unit, the superficial details of contextual applications of differentiation are stripped away to focus on abstract structures and formal conclusions. Reasoning with definitions and theorems establishes that answers and conclusions are more than conjectures; they have been analytically determined. As when students showed supporting work for answers in previous units, students will learn to present justifications for their conclusions about the behavior of functions over certain intervals or the locations of extreme values or points of inflection. The unit concludes this study of differentiation by applying abstract reasoning skills to justify solutions for realistic optimization problems.

Building the Mathematical Practices

1.E 2.D 2.E 3.E

The underlying processes of finding critical points and extrema are the foundation for the justifications students will write in this unit. Students should use calculators to graph a function and its derivatives to explore the related features of these graphs and confirm the results of their calculations.

Students often struggle with misinterpreting the characteristics of the graph of a derivative as though they are characteristics of the original function. Or, they use nonspecific language that conflates different functions (e.g., "it" rather than "f"). To prevent ongoing misconceptions, hold students accountable for extreme precision by having them practice matching graphs of functions to their derivatives and requiring them to explain their reasons to a peer.

Students also tend to rely on insufficient evidence or descriptions in their justifications, stating, for example, that "the graph of f is increasing because it's going up." This happens especially when examining derivative graphs on a calculator. Model calculus-based justifications (i.e., reasoning based on analysis of a derivative) both in discussion and in writing. Give students repeated opportunities to practice writing and revising their own justifications based on teacher feedback and feedback from their peers.

Preparing for the AP Exam

Sound reasoning must be accompanied by clear communication on the AP Exam. It may be helpful for students to use the language in the question as a starting point. Suppose a question asks, "Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer." A student who writes, "g has neither a relative maximum nor a relative minimum at $x = 10$, because . . . ," has begun well. Similarly, given a graph of the derivative, f', of a function, f, it is safer and easier for students to make arguments about f based directly on the graph of the derivative, as in, "f is concave up on $a < x < b$ because the graph of f' is increasing on $a < x < b$." Students should always refer to f, f', and f'' by name, rather than by "it" or "the function," which may leave the reader unsure of their intended meaning.

UNIT 5

Analytical Applications of Differentiation

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~15–16 CLASS PERIODS (AB) ~10–11 CLASS PERIODS (BC)
FUN-1	**5.1 Using the Mean Value Theorem**	**3.E** Provide reasons or rationales for solutions and conclusions.	
	5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points	**3.E** Provide reasons or rationales for solutions and conclusions.	
FUN-4	**5.3 Determining Intervals on Which a Function is Increasing or Decreasing**	**2.E** Describe the relationships among different representations of functions and their derivatives.	
	5.4 Using the First Derivative Test to Determine Relative (Local) Extrema	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
	5.5 Using the Candidates Test to Determine Absolute (Global) Extrema	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	5.6 Determining Concavity of Functions over Their Domains	**2.E** Describe the relationships among different representations of functions and their derivatives.	
	5.7 Using the Second Derivative Test to Determine Extrema	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
	5.8 Sketching Graphs of Functions and Their Derivatives	**2.D** Identify how mathematical characteristics or properties of functions are related in different representations.	
	5.9 Connecting a Function, Its First Derivative, and Its Second Derivative	**2.D** Identify how mathematical characteristics or properties of functions are related in different representations.	
	5.10 Introduction to Optimization Problems	**2.A** Identify common underlying structures in problems involving different contextual situations.	
	5.11 Solving Optimization Problems	**3.F** Explain the meaning of mathematical solutions in context.	
	5.12 Exploring Behaviors of Implicit Relations	**1.E** Apply appropriate mathematical rules or procedures, with and without technology. **3.E** Provide reasons or rationales for solutions and conclusions.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 5.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Suggested Activity
1	5.3	**Critique Reasoning** Arrange students in groups of four to six, provide them with a function's derivative (e.g., $g'(x) = 5x + 3$), and ask them to determine if $g(x)$ is increasing or decreasing at a specific x-value, for example, $x = -3$. Ask students to share the reasoning for their conclusion with classmates in their group. Members of the group can then provide feedback and suggestions.
2	5.4 5.7	**Think-Pair-Share** Provide students with a graph of f' and a graph of f''. Ask them to identify relative extrema and practice writing justifications for relative extrema using the first or second derivative test. Once they've written their justification, ask them to pair with a partner and share their justifications. Students can then discuss similarities or differences in their justification wording.
3	5.5	**Create a Plan** Provide students with a function represented analytically on a closed interval. Ask them to discuss and write x-values that are viable candidates for absolute extrema. Once they have established the viable candidates, ask them to design a method for analyzing the behavior of the function's graph at the candidates and for identifying the extrema.
4	5.8 5.9	**Predict and Confirm** Provide students with the graph of a differentiable function, for example, $f(x) = x^3 - 4x^2 + 4x + 1$, but do not provide the rule for the function. Ask students to sketch a graph of the derivative of the function. Once students are done, reveal the rule for $f(x)$. Ask students to calculate $f'(x)$, and use technology to graph $f'(x)$ and compare it to their sketched graph.

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

AVAILABLE RESOURCES

- Classroom Resource > **Why We Use Theorem in Calculus**
- AP Online Teacher Community Discussion > **Mean Value Existence Theorem**
- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**

TOPIC 5.1

Using the Mean Value Theorem

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.B

Justify conclusions about functions by applying the Mean Value Theorem over an interval.

ESSENTIAL KNOWLEDGE

FUN-1.B.1

If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a, b), then the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.

TOPIC 5.2

Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.C

Justify conclusions about functions by applying the Extreme Value Theorem.

ESSENTIAL KNOWLEDGE

FUN-1.C.1

If a function f is continuous over the interval (a, b), then the Extreme Value Theorem guarantees that f has at least one minimum value and at least one maximum value on (a, b).

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

AVAILABLE RESOURCES

- Classroom Resource > **Why We Use Theorem in Calculus**
- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**
- Professional Development > **Justifying Properties and Behaviors of Functions**
- Classroom Resource > **Extrema**
- **On the Role of Sign Charts in AP Calculus Exams**

TOPIC 5.3

Determining Intervals on Which a Function Is Increasing or Decreasing

SUGGESTED SKILL

Connecting Representations

2.E

Describe the relationships among different representations of functions and their derivatives.

AVAILABLE RESOURCE

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
FUN-4.A Justify conclusions about the behavior of a function based on the behavior of its derivatives.	**FUN-4.A.1** The first derivative of a function can provide information about the function and its graph, including intervals where the function is increasing or decreasing.

TOPIC 5.4

Using the First Derivative Test to Determine Relative (Local) Extrema

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.2

The first derivative of a function can determine the location of relative (local) extrema of the function.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

TOPIC 5.5

Using the Candidates Test to Determine Absolute (Global) Extrema

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.3

Absolute (global) extrema of a function on a closed interval can only occur at critical points or at endpoints.

TOPIC 5.6

Determining Concavity of Functions over Their Domains

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.4

The graph of a function is concave up (down) on an open interval if the function's derivative is increasing (decreasing) on that interval.

FUN-4.A.5

The second derivative of a function provides information about the function and its graph, including intervals of upward or downward concavity.

FUN-4.A.6

The second derivative of a function may be used to locate points of inflection for the graph of the original function.

SUGGESTED SKILL

Connecting Representations

2.E

Describe the relationships among different representations of functions and their derivatives.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Second Derivative Test Wording and Justifying Concavity Intervals**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

TOPIC 5.7

Using the Second Derivative Test to Determine Extrema

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.7

The second derivative of a function may determine whether a critical point is the location of a relative (local) maximum or minimum.

FUN-4.A.8

When a continuous function has only one critical point on an interval on its domain and the critical point corresponds to a relative (local) extremum of the function on the interval, then that critical point also corresponds to the absolute (global) extremum of the function on the interval.

TOPIC 5.8

Sketching Graphs of Functions and Their Derivatives

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.9

Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations.

FUN-4.A.10

Graphical, numerical, and analytical information from f' and f'' can be used to predict and explain the behavior of f.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCE

- Professional Development > **Justifying Properties and Behaviors of Functions**

TOPIC 5.9

Connecting a Function, Its First Derivative, and Its Second Derivative

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
FUN-4.A Justify conclusions about the behavior of a function based on the behavior of its derivatives.	**FUN-4.A.11** Key features of the graphs of f, f', and f'' are related to one another.

TOPIC 5.10

Introduction to Optimization Problems

SUGGESTED SKILL

Connecting Representations

2.A

Identify common underlying structures in problems involving different contextual situations.

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
FUN-4.B Calculate minimum and maximum values in applied contexts or analysis of functions.	**FUN-4.B.1** The derivative can be used to solve optimization problems; that is, finding a minimum or maximum value of a function on a given interval.

SUGGESTED SKILL

Justification

3.F

Explain the meaning of mathematical solutions in context.

TOPIC 5.11

Solving Optimization Problems

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function’s derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
FUN-4.C Interpret minimum and maximum values calculated in applied contexts.	**FUN-4.C.1** Minimum and maximum values of a function take on specific meanings in applied contexts.

TOPIC 5.12

Exploring Behaviors of Implicit Relations

SUGGESTED SKILLS

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
FUN-4.D Determine critical points of implicit relations.	**FUN-4.D.1** A point on an implicit relation where the first derivative equals zero or does not exist is a critical point of the function.
FUN-4.E Justify conclusions about the behavior of an implicitly defined function based on evidence from its derivatives.	**FUN-4.E.1** Applications of derivatives can be extended to implicitly defined functions. **FUN-4.E.2** Second derivatives involving implicit differentiation may be relations of x, y, and $\frac{dy}{dx}$.

THIS PAGE IS INTENTIONALLY LEFT BLANK.

AP CALCULUS AB AND BC

UNIT 6

Integration and Accumulation of Change

AP EXAM WEIGHTING
17–20% AB
17–20% BC

CLASS PERIODS
~18–20 AB
~15–16 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 6

Multiple-choice:
~25 questions (AB)
~35 questions (BC)
Free-response: 3 questions

UNIT
6

AP EXAM WEIGHTING	17–20%	AB	17–20%	BC
CLASS PERIODS	~18–20	AB	~15–16	BC

Integration and Accumulation of Change

BIG IDEA 1
Change **CHA**
- Given information about a rate of population growth over time, how can we determine how much the population changed over a given interval of time?

BIG IDEA 2
Limits **LIM**
- If compounding more often increases the amount in an account with a given rate of return and term, why doesn't compounding continuously result in an infinite account balance, all other things being equal?

BIG IDEA 3
Analysis of Functions **FUN**
- How is integrating to find areas related to differentiating to find slopes?

Developing Understanding

This unit establishes the relationship between differentiation and integration using the Fundamental Theorem of Calculus. Students begin by exploring the contextual meaning of areas of certain regions bounded by rate functions. Integration determines accumulation of change over an interval, just as differentiation determines instantaneous rate of change at a point. Students should understand that integration is a limiting case of a sum of products (areas) in the same way that differentiation is a limiting case of a quotient of differences (slopes). Future units will apply the idea of accumulation of change to a variety of realistic and geometric applications.

Building the Mathematical Practices

1.D **1.E** **2.C** **3.D**

Students often struggle with the relationship between differentiation and integration. They think that integration is simply differentiation in reverse order. However, to apply the rules of integration correctly, students must think more strategically, taking into consideration how the "pieces" fit together. Students will need explicit guidance for choosing an appropriate antidifferentiation strategy that's based on the underlying patterns in different categories of integrands (e.g., using u-substitution when they recognize that the integrand is a factor of the derivative of a composite function or using integration by parts for an integrand, udv, that is related to a term in the derivative of the product uv **BC ONLY**).

Students also struggle with relating a symbolic limit of a Riemann sum to that limit expressed as a definite integral, because of the complexity of the expressions. To help students feel more comfortable working with these expressions, use explicit strategies, such as helping students to break complex expressions into familiar components, or matching expressions for a definite integral with the limit of a Riemann sum, and vice versa.

Preparing for the AP Exam

Students should be careful applying the chain rule, both when differentiating functions defined by integrals and when integrating using u-substitution. Students will need to recognize integrands that are factors of a chain rule derivative and should practice u-substitution until the process is internalized. Students will additionally need to recognize integrands that suggest strategies such as integration by parts or partial fractions and should use mixed practice in preparation for the exam **BC ONLY**.

When using a calculator to evaluate a definite integral in a free-response question, students should present the expression for the definite integral, including endpoints of integration, and an appropriately placed differential. When evaluating an integral without a calculator, students should present an appropriate antiderivative; they should include a constant of integration with indefinite integrals. As always, students should be careful about parentheses usage and should avoid writing strings of equal signs equating expressions that are not equal.

UNIT
6

Integration and Accumulation of Change

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods
			~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
CHA-4	**6.1 Exploring Accumulations of Change**	**4.B** Use appropriate units of measure.	
LIM-5	**6.2 Approximating Areas with Riemann Sums**	**1.F** Explain how an approximated value relates to the actual value.	
	6.3 Riemann Sums, Summation Notation, and Definite Integral Notation	**2.C** Identify a re-expression of mathematical information presented in a given representation.	
FUN-5	**6.4 The Fundamental Theorem of Calculus and Accumulation Functions**	**1.D** Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	6.5 Interpreting the Behavior of Accumulation Functions Involving Area	**2.D** Identify how mathematical characteristics or properties of functions are related in different representations.	
FUN-6	**6.6 Applying Properties of Definite Integrals**	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
	6.7 The Fundamental Theorem of Calculus and Definite Integrals	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
	6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation	**4.C** Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	6.9 Integrating Using Substitution	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	6.10 Integrating Functions Using Long Division and Completing the Square	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
FUN-6	**6.11** Integrating Using Integration by Parts **BC ONLY**	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	6.12 Integrating Using Linear Partial Fractions **BC ONLY**	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
LIM-6	**6.13** Evaluating Improper Integrals **BC ONLY**	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-6	**6.14** Selecting Techniques for Antidifferentiation	**1.C** Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 6. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	6.3	**Quickwrite** Present the class with several examples of definite integrals set equal to Riemann sums in summation notation, for example $\int_{-2}^{5}(x^2+5)dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{7}{n}\right)\left(\left(-2+\frac{7}{n}i\right)^2+5\right)$. Ask students to take five minutes to identify and write about all common elements between the two expressions and why they think the two expressions are equivalent. After finishing the five minutes, ask students to share their observations with the class.
2	6.9	**Look For a Pattern** Present students with several indefinite integrals and proposed, yet incorrect, antiderivatives, for example, $\int(5x+2)^{20}\,dx=\frac{1}{21}(5x+2)^{21}+C$. Ask them to check the antiderivatives by differentiating each and comparing to the original integrands. As students see that each antiderivative is incorrect, ask them to identify a pattern within the errors. Identifying this pattern will establish the foundation for integrating using substitution.
3	6.10	**Odd One Out** To help students select a strategy, form groups of four, presenting each student an indefinite integral whose integrand is rational. For each group, include one integrand that requires long division or completing the square. Ask students to decide if their example fits with the group. Identifying the odd one out will help students connect integrand form to the appropriate strategy.

TOPIC 6.1

Exploring Accumulations of Change

SUGGESTED SKILL

Communication and Notation

4.B

Use appropriate units of measure.

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.A

Interpret the meaning of areas associated with the graph of a rate of change in context.

ESSENTIAL KNOWLEDGE

CHA-4.A.1

The area of the region between the graph of a rate of change function and the x axis gives the accumulation of change.

CHA-4.A.2

In some cases, accumulation of change can be evaluated by using geometry.

CHA-4.A.3

If a rate of change is positive (negative) over an interval, then the accumulated change is positive (negative).

CHA-4.A.4

The unit for the area of a region defined by rate of change is the unit for the rate of change multiplied by the unit for the independent variable.

SUGGESTED SKILL

Implementing Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

AVAILABLE RESOURCE

- Classroom Resource > **Reasoning from Tabular Data**

TOPIC 6.2

Approximating Areas with Riemann Sums

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE

LIM-5.A

Approximate a definite integral using geometric and numerical methods.

ESSENTIAL KNOWLEDGE

LIM-5.A.1

Definite integrals can be approximated for functions that are represented graphically, numerically, analytically, and verbally.

LIM-5.A.2

Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or nonuniform partitions.

LIM-5.A.3

Definite integrals can be approximated using numerical methods, with or without technology.

LIM-5.A.4

Depending on the behavior of a function, it may be possible to determine whether an approximation for a definite integral is an underestimate or overestimate for the value of the definite integral.

TOPIC 6.3

Riemann Sums, Summation Notation, and Definite Integral Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
LIM-5.B Interpret the limiting case of the Riemann sum as a definite integral.	**LIM-5.B.1** The limit of an approximating Riemann sum can be interpreted as a definite integral. **LIM-5.B.2** A Riemann sum, which requires a partition of an interval I, is the sum of products, each of which is the value of the function at a point in a subinterval multiplied by the length of that subinterval of the partition.
LIM-5.C Represent the limiting case of the Riemann sum as a definite integral.	**LIM-5.C.1** The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where n is the number of subintervals, Δx_i is the width of the ith subinterval, and x_i^* is a value in the ith subinterval. **LIM-5.C.2** A definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "Say" Some of the Notation**
- AP Online Teacher Community Discussion > **Definite Integral as the Limit of a Riemann Sum**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

ILLUSTRATIVE EXAMPLES

For FUN-5.A.1:

$f(x) = \int_0^x e^{-t^2}\,dt.$

AVAILABLE RESOURCES

- Professional Development > **The Fundamental Theorem of Calculus**
- Classroom Resource > **Functions Defined by Integrals**

TOPIC 6.4

The Fundamental Theorem of Calculus and Accumulation Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-5

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE

FUN-5.A

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE

FUN-5.A.1

The definite integral can be used to define new functions.

FUN-5.A.2

If f is a continuous function on an interval containing a, then $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$, where x is in the interval.

TOPIC 6.5

Interpreting the Behavior of Accumulation Functions Involving Area

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING

FUN-5

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE

FUN-5.A

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE

FUN-5.A.3

Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as $g(x)=\int_a^x f(t)dt$.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 6.6

Applying Properties of Definite Integrals

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.A

Calculate a definite integral using areas and properties of definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.A.1

In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.

FUN-6.A.2

Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.

FUN-6.A.3

The definition of the definite integral may be extended to functions with removable or jump discontinuities.

TOPIC 6.7

The Fundamental Theorem of Calculus and Definite Integrals

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- Professional Development > **The Fundamental Theorem of Calculus**

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.B

Evaluate definite integrals analytically using the Fundamental Theorem of Calculus.

ESSENTIAL KNOWLEDGE

FUN-6.B.1

An antiderivative of a function f is a function g whose derivative is f.

FUN-6.B.2

If a function f is continuous on an interval containing a, the function defined by $F(x)=\int_a^x f(t)\,dt$ is an antiderivative of f for x in the interval.

FUN-6.B.3

If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

SUGGESTED SKILL

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

TOPIC 6.8

Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.C

Determine antiderivatives of functions and indefinite integrals, using knowledge of derivatives.

ESSENTIAL KNOWLEDGE

FUN-6.C.1

$\int f(x)dx$ is an indefinite integral of the function f and can be expressed as $\int f(x)dx = F(x)+C$, where $F'(x)= f(x)$ and C is any constant.

FUN-6.C.2

Differentiation rules provide the foundation for finding antiderivatives.

FUN-6.C.3

Many functions do not have closed-form antiderivatives.

TOPIC 6.9

Integrating Using Substitution

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.D

For integrands requiring substitution or rearrangements into equivalent forms:

(a) Determine indefinite integrals.

(b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.D.1

Substitution of variables is a technique for finding antiderivatives.

FUN-6.D.2

For a definite integral, substitution of variables requires corresponding changes to the limits of integration.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Applying Procedures for Integration by Substitution**
- AP Online Teacher Community Discussion > **U-Substitution with Improper Integrals**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

TOPIC 6.10

Integrating Functions Using Long Division and Completing the Square

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.D

For integrands requiring substitution or rearrangements into equivalent forms:

(a) Determine indefinite integrals.

(b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.D.3

Techniques for finding antiderivatives include rearrangements into equivalent forms, such as long division and completing the square.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

TOPIC 6.14

Selecting Techniques for Antidifferentiation

This topic is intended to focus on the skill of selecting an appropriate procedure for antidifferentiation. Students should be given opportunities to practice when and how to apply all learning objectives relating to antidifferentiation.

AP CALCULUS AB AND BC

UNIT 7

Differential Equations

AP EXAM WEIGHTING	6–12% AB 6–9% BC	
CLASS PERIODS	~8–9 AB ~9–10 BC	

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 7

Multiple-choice:
~15 questions (AB)
~20 questions (BC)
Free-response: 3 questions

UNIT
7

AP EXAM WEIGHTING	6–12% AB	6–9% BC
CLASS PERIODS	~8–9 AB	~9–10 BC

Differential Equations

BIG IDEA 3
Analysis of Functions **FUN**

- How can we derive a model for the number of computers, C, infected by a virus, given a model for how fast the computers are being infected, $\frac{dC}{dt}$, at a particular time?

Developing Understanding

In this unit, students will learn to set up and solve separable differential equations. Slope fields can be used to represent solution curves to a differential equation and build understanding that there are infinitely many general solutions to a differential equation, varying only by a constant of integration. Students can locate a unique solution relevant to a particular situation, provided they can locate a point on the solution curve. By writing and solving differential equations leading to models for exponential growth and decay and logistic growth **BC ONLY**, students build understanding of topics introduced in Algebra II and other courses.

Building the Mathematical Practices

1.E **2.C** **3.G** **4.D**

In this unit, students will translate mathematical information from one representation to another by matching equations and slope fields, rewriting verbal statements as differential equations, and sketching slope fields that match their symbolic representations. Provide students with explicit guidance on how to select an appropriate graphing technique. As students practice Euler's method, encourage them to transfer skills using tangent line approximations, rather than simply memorizing an algorithm **BC ONLY**.

Because the problems in this unit model real-world scenarios, help students to develop proficiency in transferring the mathematical procedures they've learned in "x's and y's" to equivalent environments with variable names other than x, y, and t. Using differentiation to confirm that solutions to differential equations are accurate and appropriate also helps students to develop an understanding of what it means to say that an equation is a solution to a differential equation.

Preparing for the AP Exam

Students should practice setting up and solving contextual questions involving separable differential equations until the solution strategy becomes routine: separate variables, antidifferentiate both sides of the equation and add a constant of integration, use initial conditions to determine the constant of integration, and rearrange the resulting expression to complete the solution. Failure to separate variables or omitting the constant of integration severely limits the number of points a student can earn on the AP Exam. A common error in antidifferentiation is to assume that all differential equations involving fractions have logarithmic solutions, presumably because some do.

Students should learn to recognize the forms of differential equations resulting in exponential and logistic **BC ONLY** models. These may be used or interpreted without performing the derivation. Students should also be reminded that differential equations give us information about the derivative and may be used directly to find information about a slope or rate of change.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~8–9 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
FUN-7	**7.1 Modeling Situations with Differential Equations**	**2.C** Identify a re-expression of mathematical information presented in a given representation.	
	7.2 Verifying Solutions for Differential Equations	**3.G** Confirm that solutions are accurate and appropriate.	
	7.3 Sketching Slope Fields	**2.C** Identify a re-expression of mathematical information presented in a given representation.	
	7.4 Reasoning Using Slope Fields	**4.D** Use appropriate graphing techniques.	
	7.5 Approximating Solutions Using Euler's Method **BC ONLY**	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	7.6 Finding General Solutions Using Separation of Variables	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	7.7 Finding Particular Solutions Using Initial Conditions and Separation of Variables	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	7.8 Exponential Models with Differential Equations	**3.G** Confirm that solutions are accurate and appropriate.	
	7.9 Logistic Models with Differential Equations **BC ONLY**	**3.F** Explain the meaning of mathematical solutions in context.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 7.
Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	7.3	**Match Mine** Give student pairs a blank 3 × 3 game board and nine graphs of slope fields, each on a separate card. Some should be in terms of x only, some in terms of y only, and some in terms of x and y. Be sure to include at least one trigonometric function. Student A arranges the graphs on the grid without showing Student B and then describes the arrangement so Student B can try to match it on their own board.
2	7.6	**Numbered Heads Together** Have each student complete the same problem individually (e.g., $y' = 2xy^2$, $\frac{dy}{dx} = y^2 + 1$, or $3y\,dy = (x^2 + 1)dx$). Make sure to use a variety of notation in whatever problem you pick. Then have students compare answers and procedures within groups. Students fix any mistakes until they all agree on the same answer.
3	7.7 7.8	**Collaborative Poster** Assign each student a role within their group: • Separating the variables • Integrating both sides • Finding C • Writing the final particular solution Then distribute a free-response question to each group and have them work on their assigned roles to solve the problem together. Examples include the following: • **2002 Form B #5(b)** (not transcendental) • **2011 #5(c)** (transcendental) • **2012 #5(c)** (transcendental) • **2014 #6(c)** (transcendental)

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

TOPIC 7.1

Modeling Situations with Differential Equations

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.A

Interpret verbal statements of problems as differential equations involving a derivative expression.

ESSENTIAL KNOWLEDGE

FUN-7.A.1

Differential equations relate a function of an independent variable and the function's derivatives.

TOPIC 7.2

Verifying Solutions for Differential Equations

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.B

Verify solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.B.1

Derivatives can be used to verify that a function is a solution to a given differential equation.

FUN-7.B.2

There may be infinitely many general solutions to a differential equation.

SUGGESTED SKILL

Justification

3.G

Confirm that solutions are accurate and appropriate.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Classroom Resource > **Slope Fields**
- Classroom Resource > **Differential Equations**

TOPIC 7.3

Sketching Slope Fields

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.C

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.C.1

A slope field is a graphical representation of a differential equation on a finite set of points in the plane.

FUN-7.C.2

Slope fields provide information about the behavior of solutions to first-order differential equations.

TOPIC 7.4

Reasoning Using Slope Fields

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.C

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.C.3

Solutions to differential equations are functions or families of functions.

SUGGESTED SKILL

Communication and Notation

4.D

Use appropriate graphing techniques.

AVAILABLE RESOURCES

- Classroom Resource > **Slope Fields**
- Classroom Resource > **Differential Equations**

TOPIC 7.6

Finding General Solutions Using Separation of Variables

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.D

Determine general solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.D.1

Some differential equations can be solved by separation of variables.

FUN-7.D.2

Antidifferentiation can be used to find general solutions to differential equations.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Classroom Resource > **Differential Equations**

TOPIC 7.7

Finding Particular Solutions Using Initial Conditions and Separation of Variables

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.E

Determine particular solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.E.1

A general solution may describe infinitely many solutions to a differential equation. There is only one particular solution passing through a given point.

FUN-7.E.2

The function F defined by $F(x) = y_0 + \int_a^x f(t)\,dt$ is a particular solution to the differential equation $\frac{dy}{dx} = f(x)$, satisfying $F(a) = y_0$.

FUN-7.E.3

Solutions to differential equations may be subject to domain restrictions.

TOPIC 7.8

Exponential Models with Differential Equations

SUGGESTED SKILL

Justification

3.G

Confirm that solutions are accurate and appropriate.

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
FUN-7.F Interpret the meaning of a differential equation and its variables in context.	**FUN-7.F.1** Specific applications of finding general and particular solutions to differential equations include motion along a line and exponential growth and decay. **FUN-7.F.2** The model for exponential growth and decay that arises from the statement “The rate of change of a quantity is proportional to the size of the quantity” is $\frac{dy}{dt} = ky$.
FUN-7.G Determine general and particular solutions for problems involving differential equations in context.	**FUN-7.G.1** The exponential growth and decay model, $\frac{dy}{dt} = ky$, with initial condition $y = y_0$ when $t = 0$, has solutions of the form $y = y_0 e^{kt}$.

AP CALCULUS AB AND BC

UNIT 8

Applications of Integration

AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~19–20 AB	~13–14 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 8

Multiple-choice: ~30 questions
Free-response: 3 questions

UNIT
8

AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~19–20 AB	~13–14 BC

Applications of Integration

BIG IDEA 1
***Change* CHA**

- How is finding the number of visitors to a museum over an interval of time based on information about the rate of entry similar to finding the area of a region between a curve and the x-axis?

Developing Understanding

In this unit, students will learn how to find the average value of a function, model particle motion and net change, and determine areas, volumes, and lengths **BC ONLY** defined by the graphs of functions. Emphasis should be placed on developing an understanding of integration that can be transferred across these and many other applications. Understanding that the area, volume, and length **BC ONLY** problems studied in this unit are limiting cases of Riemann sums of rectangle areas, prism volumes, or segment lengths **BC ONLY** saves students from memorizing a long list of seemingly unrelated formulas and develops meaningful understanding of integration.

Building the Mathematical Practices

1.D **2.D** **3.D** **4.C**

As in Unit 4, students will need to practice interpreting verbal scenarios, extracting relevant mathematical information, selecting an appropriate procedure, and then applying that procedure correctly and interpreting their solution in the context of the problem. Now that students have been exposed to application problems involving both differentiation and antidifferentiation, some may struggle to determine which procedure is applicable. Walk students through different types of scenarios and explain the underlying reasons why some situations call for differentiation while others call for integration.

This unit also involves geometric applications of integration. When using the disc and washer methods, focusing on orientation (i.e., horizontal or vertical) will help students determine whether the "thickness" is with respect to x or y. Students should practice solving variations on these calculus-based geometry problems until they can decide which variable to integrate with respect to without prompting. Relating graphical representations to symbolic representations, such as Riemann sums and definite integrals, develops these skills and helps students to master the content.

Preparing for the AP Exam

On the AP Exam, students need to identify relevant information conveyed in various representations. Key words, such as "accumulation" or "net change," help to identify mathematical structures and corresponding solution strategies. Some students confuse the average value and the average rate of change of a function on an interval. To alleviate confusion, first provide students with average value problems accompanied by relevant graphs and guide them to an understanding of why an average value may be less than, equal to, or greater than the midpoint of the range. Then review average rate of change problems from Unit 2 and present students with free-response questions that will allow them to practice distinguishing between average value and average rate of change problems.

In free-response questions, continue to require students to show supporting work by presenting a correct expression using appropriate notation and the mathematical structures of solutions, as in $V = \pi \int_1^4 \left[\left(f(x) - 3 \right)^2 - \left(g(x) - 3 \right)^2 \right] dx$, for example.

UNIT
8

Applications of Integration

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-4	**8.1 Finding the Average Value of a Function on an Interval**	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	8.2 Connecting Position, Velocity, and Acceleration of Functions Using Integrals	**1.D** Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
CHA-5	**8.4 Finding the Area Between Curves Expressed as Functions of x**	**4.C** Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	8.5 Finding the Area Between Curves Expressed as Functions of y	**1.E** Apply appropriate mathematical rules or procedures, with and without technology.	
	8.6 Finding the Area Between Curves That Intersect at More Than Two Points	**2.B** Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.	
	8.7 Volumes with Cross Sections: Squares and Rectangles	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
	8.8 Volumes with Cross Sections: Triangles and Semicircles	**3.D** Apply an appropriate mathematical definition, theorem, or test.	
	8.9 Volume with Disc Method: Revolving Around the x- or y-Axis	**3.D** Apply an appropriate mathematical definition, theorem, or test.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-5	**8.10 Volume with Disc Method: Revolving Around Other Axes**	**2.D** Identify how mathematical characteristics or properties of functions are related in different representations.	
	8.11 Volume with Washer Method: Revolving Around the x- or y-Axis	**4.E** Apply appropriate rounding procedures.	
	8.12 Volume with Washer Method: Revolving Around Other Axes	**2.D** Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-6	**8.13 The Arc Length of a Smooth, Planar Curve and Distance Traveled** **BC ONLY**	**3.D** Apply an appropriate mathematical definition, theorem, or test.	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 8. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	8.1	**Scavenger Hunt** Post around the room 8–10 problem cards, each of which also includes a solution to a previous problem. Include average value problems with tables and functions; also include tables that have values from 0 to 16, for example, but where the average value is only on the interval from 0 to 12. Card 1 could say: Find the average value of $f(x) = \sin(x)$ on the interval $[0, \pi]$. A second card would have the answer to that card $\left(\frac{2}{\pi}\right)$ along with a new question: Find the average value of $f(x) = 3x^2 - 3$ on $[1, 3]$. A third card would have that answer (10) and so on. The answer to the last card goes on top of the first card.
2	8.6	**Round Table** In groups of four, each student has an identical paper with the same free-response question (e.g., **2015 AB #2(a)**), along with four labeled boxes representing steps in the problem: ▪ Identify all points of intersection. ▪ Set up the integral(s). ▪ Integrate by hand. ▪ Integrate using a calculator. Have students complete the first step on their paper, and then pass the paper clockwise to another member in their group. That student checks the first step and then completes the second step on the paper. Students rotate again and the process continues until each student has their own paper back.
3	8.9 8.10 8.11 8.12	**Quiz-Quiz-Trade** Create cards with problems revolving around the x- or y-axis and others revolving around other axes (e.g., $y = x$ or $y = 3$). Give each student a card and have them write their answer on the back. Students quiz a partner about their own card then switch cards and repeat the process with a new partner. For the first round, concentrate on just setting up the integrals (e.g., **2009 AB Form B #4(c)**, **2010 AB/BC #4(b)**, **2011 AB #3(c)**, and **2013 AB #5(b)**). In the second round, students can use their calculators to find the volume (e.g., **2001 AB #1(c)**, **2006 AB/BC #1(b)**, **2007 AB/BC #1(b)**, and **2008 AB Form B #1(b)**).

TOPIC 8.1

Finding the Average Value of a Function on an Interval

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.B

Determine the average value of a function using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-4.B.1

The average value of a continuous function f over an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx.$

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCE

- Classroom Resource > **Motion**

TOPIC 8.2

Connecting Position, Velocity, and Acceleration of Functions Using Integrals

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.C

Determine values for positions and rates of change using definite integrals in problems involving rectilinear motion.

ESSENTIAL KNOWLEDGE

CHA-4.C.1

For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.

TOPIC 8.3

Using Accumulation Functions and Definite Integrals in Applied Contexts

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE	ESSENTIAL KNOWLEDGE
CHA-4.D Interpret the meaning of a definite integral in accumulation problems.	**CHA-4.D.1** A function defined as an integral represents an accumulation of a rate of change. **CHA-4.D.2** The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.
CHA-4.E Determine net change using definite integrals in applied contexts.	**CHA-4.E.1** The definite integral can be used to express information about accumulation and net change in many applied contexts.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

SUGGESTED SKILL

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

TOPIC 8.4

Finding the Area Between Curves Expressed as Functions of x

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.1

Areas of regions in the plane can be calculated with definite integrals.

TOPIC 8.5

Finding the Area Between Curves Expressed as Functions of y

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.2

Areas of regions in the plane can be calculated using functions of either x or y.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, symbolic, numerical, and/or verbal representations.

TOPIC 8.6

Finding the Area Between Curves That Intersect at More Than Two Points

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.3

Areas of certain regions in the plane may be calculated using a sum of two or more definite integrals or by evaluating a definite integral of the absolute value of the difference of two functions.

TOPIC 8.7

Volumes with Cross Sections: Squares and Rectangles

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.B

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.B.1

Volumes of solids with square and rectangular cross sections can be found using definite integrals and the area formulas for these shapes.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

ILLUSTRATIVE EXAMPLES

- Illustrative examples of other cross sections in CHA-5.B.3:
- * The volume of a funnel whose cross sections are circles can be found using the area formula for a circle and definite integrals (see **2016 AB Exam FRQ #5(b)**).
- * The volume of a solid whose cross sectional area is defined using a function can be found using the known area function and a definite integral (see **2009 AB Exam FRQ #4(c)**).

TOPIC 8.8

Volumes with Cross Sections: Triangles and Semicircles

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.B

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.B.2

Volumes of solids with triangular cross sections can be found using definite integrals and the area formulas for these shapes.

CHA-5.B.3

Volumes of solids with semicircular and other geometrically defined cross sections can be found using definite integrals and the area formulas for these shapes.

TOPIC 8.9

Volume with Disc Method: Revolving Around the *x*- or *y*-Axis

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.1

Volumes of solids of revolution around the x- or y-axis may be found by using definite integrals with the disc method.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

TOPIC 8.10

Volume with Disc Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.2

Volumes of solids of revolution around any horizontal or vertical line in the plane may be found by using definite integrals with the disc method.

TOPIC 8.11

Volume with Washer Method: Revolving Around the x- or y-Axis

SUGGESTED SKILL

Communication and Notation

4.E

Apply appropriate rounding procedures.

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.3

Volumes of solids of revolution around the x- or y-axis whose cross sections are ring shaped may be found using definite integrals with the washer method.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCE

- Classroom Resource > **Volumes of Solids of Revolution**

TOPIC 8.12

Volume with Washer Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.4

Volumes of solids of revolution around any horizontal or vertical line whose cross sections are ring shaped may be found using definite integrals with the washer method.